Transition Dynamics in Equilibrium Search*

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Abstract

We study a dynamic equilibrium search model where sellers differ in their urgency to liquidate an asset. Buyers strategically choose which price to offer, but do not know a given seller's type. We study the behavior of transaction prices, seller composition, and search duration throughout the equilibrium transition path. Generically, the transition includes a period during which all buyers offer a single price, even when price dispersion occurs in steady state. Moreover, prices often overshoot en route to their steady state level, rather than monotonically converging. In extensions to our model, prices can even oscillate around the steady state. These dynamics provide insight into fire sales, such as home foreclosures, which we illustrate by examining specific shocks and the transitions which ensue.

Keywords: Equilibrium search, transition paths, dynamic price formation, seller composition, motivated sellers, fire sales, liquidity

JEL Classification: D83, C73

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1 Introduction

Sellers often differ in their urgency to liquidate an asset. In real estate markets, buyers frequently speculate as to how "motivated" the seller is while selecting a price to offer. A "low ball" offer can save substantial money but has a lower probability of being accepted. This success rate crucially depends on the proportion of sellers who are motivated, but future market conditions are equally important. Even a motivated seller will turn down low offers if full-price offers are likely to arrive shortly after.

To address future market conditions, equilibrium search theory has primarily focused on steady state analysis. New entrants of each type precisely replace those exiting the market, which holds the current success rate and distribution of future offers constant. This convenient assumption provides tractability, and is usually interpreted as a long-run outcome of a market. However, we are frequently interested in the short-run impact of market shocks. For instance, the housing market could be temporarily flooded with a number of foreclosed homes. Eventually, the market can absorb these foreclosures and return to its steady state; but in the meantime, market turmoil is evident in reduced transaction volumes, collapsed prices, and lengthy listing periods. Steady state analysis alone is insufficient to predict the duration of these painful episodes or evaluate policy proposals to address them.

In this paper, we analyze transition dynamics in an equilibrium search environment for a durable asset. In our baseline model, sellers enter the market as either relaxed or desperate, where the latter incurs a cost of holding the asset, representing a greater need for liquidity. The type of a seller is private information. On randomly meeting a seller, homogenous buyers set a price for a take-it-or-leave-it offer. If the seller accepts the offer, both the buyer and the seller exit the market. Rejected offers cannot be recalled.

For any initial population of relaxed and desperate sellers, we characterize the equilibrium transition path as the market approaches steady state. At each instant, all participants correctly anticipate the future equilibrium states of the market, such as changes in the distribution of price offers. Indeed, all participants know that prices will eventually reach their steady state levels; yet even when the current price differs from this long run level, infrequent matches prevent anyone from performing arbitrage. Thus, due to search frictions, adjustment takes time even when the path is common knowledge.

The dynamic transitions reveal interesting behavior. First, the unique equilibrium transition generically begins with a degenerate price distribution, even when the steady state involves price dispersion. Even small imbalances in the mixture of relaxed and desperate sellers make it optimal for buyers to exclusively target the relatively-abundant segment of the market. At the same time, this collapse of the price distribution gradually corrects the imbalance, since the abundant targeted sellers will sell their asset and exit at a faster rate. When dispersed pricing becomes optimal again, it generates discontinuous jumps in prices.

Second, prices often do not follow a monotonic path toward steady state. For instance, a temporary surge in the population of desperate sellers will cause a discrete drop in market prices, but the recovery that follows will overshoot then slowly return to steady state. This non-monotonic behavior occurs because a portion of the transition requires a degenerate price distribution. For instance, by exclusively targeting desperate sellers, buyers reduce their stock but unintentionally build the stock of relaxed sellers; the subsequent resolution of this imbalance pushes prices in the opposite direction. This overshooting of the steady state price is exacerbated in an extension to the model in which relaxed sellers may randomly become desperate. There, prices can oscillate while converging to steady state.

We illustrate potential uses of the model by considering several experiments in which initial conditions or model parameters unexpectedly change, tracing the resulting transition that follows the shock. The model has particular relevance to the theory of fire sales, meaning the forced sale of an asset at a dislocated price, well below its fundamental value (Schleifer and Vishny, 2011).[1] The concept of a fire sale readily suggests the need for a search model, since it posits that better price offers exist but are not readily encountered. Yet fire sales are also temporary phenomenon, not permanent conditions. The loss of dispersion in our transition path is consistent with the temporary freezing of a portion of the market after a negative shock.

We contribute to an emerging branch of the equilibrium search literature, examining dynamics when a market is not in steady state.[2] Transitions are studied in some money search models,[3] but these generate a single price at each point. An exception featuring price dispersion is the simultaneous search model of Burdett, *et al* (2017), in which money holders receive a random number of price quotes, inducing competition among sellers and a distribution of prices. They provide an example (in Section 10) where price dispersion persists through the transition path because multiple quotes are exogenously given. In our price-posting setting with a dispersed steady state, however, we learn that a shock will collapse and later restore the price distribution in almost every transition.

Recurring cycles, in which the populations and prices endlessly repeat a given sequence, can occur in some money search models (Boldrin, *et al*, 1993; Coles and Wright, 1998; Burdett, *et al*, 2017) and the consumer search model of Albrecht, *et*

[1]For example, when a bank or homeowner liquidates an asset to cover short-term borrowing costs, Campbell, *et al* (2011) report that the asset sells at a 27% discount relative to market value. When airlines sell airplanes to pay back loans, they are discounted 10-20% according to Pulvino (1998).

[2]Diamond (1987) and Albrecht, *et al* (2007) provide the closest model to our steady state analysis, with price posting in the former and bargaining over prices in the latter.

[3]*e.g.* Boldrin, *et al* (1993); Coles and Wright (1998). Lagos, *et al* (2017) offer a thorough survey of the money search literature.

al (2013). The latter model features a discrete-time price-posting environment with two types of consumers differing in their willingness to pay. In equilibrium, when the market exclusively offers high prices, low-type customers accumulate. Eventually it becomes profitable to offer a low price for one period to clear the market, but sellers then revert to a high price and repeat the cycle. In our model, the targeted group does change across different phases on the transition path, but this cannot occur in a repeated cycle.[4] Moreover, our interest lies in the market's transitional dynamics from an arbitrary starting position back to steady state after the economy is hit with a temporary or a permanent shock.

Second, our model highlights the importance of search frictions in general and price dispersion in particular as a means of propagating a market shock over time.[5] Infrequent trading opportunities prevent immediate adjustment of market prices by limiting the potential for arbitrage. In a housing model without price dispersion, Díaz and Jerez (2013) note that unsuccessful sales in one period add excess sellers to the next period, leading to further propagation of low prices. This feature takes on added nuance in an environment with potential price dispersion. The price distribution affects the overall number of sellers remaining in the market, but it also determines which types remain. The levels and proportions of seller types have distinct steady state requirements, making it possible to satisfy one while violating the other, and thus prolong the recovery from a shock. This is precisely what leads to overshooting in our baseline model and oscillating prices in our extension, propagating the shock

[4] This is ruled out because continuity of the Bellman equations results in continuous prices over time. A desperate phase cannot jump to a relaxed phase, for instance, because prices are continuously falling in the former and continuously rising in the latter. This pushes either phase toward the dispersed path, and make it unprofitable to offer only one price thereafter. Even in our extended model, the oscillations dampen toward the steady state rather than sustaining a true cycle.

[5] A related work is Guren and McQuade (2013), which examines how foreclosures exacerbate downturns in the housing market; however, equilibrium prices in their model exhibit monotonic transition.

even further.

2 Baseline Model

Consider a continuous time environment, with infinitesimal buyers and sellers. There are two types of sellers: relaxed (entering the market at rate η) and desperate (entering at rate δ). Either type only exits after selling their single unit of an indivisible good, and seller type is private information. The measure of relaxed sellers in the market at time t is $h_r(t)$ and desperate sellers is $h_d(t)$. For notational simplicity, we omit the function of time and use Newton's notation for time derivatives.

Sellers encounter ex-ante homogeneous potential buyers at exogenous rate λ, whereupon the buyer makes a take-it-or-leave-it offer; if rejected, both parties continue their search, with no recall of past offers. Time is discounted at rate ρ. We assume that $\lambda > \rho$ and $\rho < 1$ to make search productive given its time cost.

2.1 Buyers

Upon purchasing the asset, a buyer enjoys instantaneous value y from the good perpetually, with a present discounted value of $\frac{y}{\rho}$. On meeting a seller, the buyer must decide whether to offer price p_d, which only desperate sellers are willing to accept, or price p_r, which any seller will accept. Since he cannot distinguish seller types, he faces the tradeoff that the former will cost him less but is less likely to be accepted. Specifically, the fraction who would accept the lower price is denoted:

$$\phi \equiv \frac{h_d}{h_r + h_d}. \tag{1}$$

The relative gain from targeting the desperate sellers (by offering p_d) is expressed as:

$$\Pi \equiv \phi \left(\frac{y}{\rho} - p_d \right) - \frac{y}{\rho} + p_r, \tag{2}$$

since the p_d offer is only accepted with probability ϕ while the offer p_r is accepted for sure. If indifferent, meaning $\Pi = 0$, the buyer can employ a mixed strategy[6] offering p_d with probability μ. Individual rationality requires that $\mu = 1$ if $\Pi > 0$ and $\mu = 0$ if $\Pi < 0$.

We note that in this baseline model, buyers do not make inter-temporal decisions; but this is merely for analytic convenience. In Section 5.1, we extend the model to allow strategic entry and exit of buyers over time, making the match rate λ endogenous. The resulting behavior is highly similar, as discussed later.

2.2 Sellers

The asset provides income to the seller until it is sold: relaxed sellers collect x per unit of time,[7] while desperate sellers collect $x - c$, where $c > 0$. We assume that $y \geq x$, which indicates that buyers value the asset more than any seller, making all transactions efficient. Indeed, if $y < x$, then no buyer would offer a price that relaxed sellers would accept, effectively excluding relaxed sellers from the market.

Let V_d denote the present value of expected utility for a desperate seller:

$$\rho V_d = x - c + \dot{V}_d + \lambda \left[\mu \left(p_d - V_d \right) + (1 - \mu) \left(p_r - V_d \right) \right]. \tag{3}$$

[6]This can be interpreted as randomization by the individual buyer, or as a fraction of the buyer population who always offers the desperate price.

[7]Recall that the asset is homogenous from the perspective of the buyers; so any difference in the asset value is idiosyncratic to the seller's needs (such as selling a home to move to a new city, or needing liquidity for other purchases), rather than the asset's fundamental productivity.

These desperate sellers earn $x - c$ from the asset each unit of time, and encounter buyers at rate λ. The $\dot{V}_d$ term captures any change in the future value of search, anticipating changes in μ, p_d, or p_r. Note that desperate sellers are willing to accept either price. The equilibrium desperate price makes the desperate seller indifferent between accepting the offer or continuing her search:

$$p_d = V_d. \tag{4}$$

The present value of expected utility for a relaxed seller, V_r, is:

$$\rho V_r = x + \dot{V}_r + \lambda(1 - \mu)\left(p_r - V_r\right). \tag{5}$$

Relaxed sellers earn x from the asset each unit of time, and while they encounter buyers at the same rate λ, they reject all desperate offers. The equilibrium price will push the relaxed sellers to indifference:

$$p_r = V_r. \tag{6}$$

The key difference between Eqs. 3 and 5 is that relaxed sellers avoid the cost c, which ensures that $V_r > V_d$. Both are free to accept either offer, but since $p_r > p_d$, it is always optimal for relaxed sellers to reject desperate price offers.

2.3 Population Dynamics

The fraction of desperate sellers in the market is particularly important in this model, since this becomes the probability with which p_d offers are accepted. We thus track how the stock of sellers in the market adjusts over time. First, consider the population of desperate sellers. These enter the market at rate δ, but they accept any offer, and thus exit the market at rate λh_d. Thus, the net change in their population is:

$$\dot{h}_d = \delta - \lambda h_d. \tag{7}$$

Relaxed sellers, on the other hand, enter the market at rate η, but only accept relaxed price offers, exiting at rate $\lambda(1-\mu)$. Thus:

$$\dot{h}_r = \eta - \lambda(1-\mu)h_r. \tag{8}$$

2.4 Equilibrium Definition

For a given initial seller population $h_r(0)$ and $h_d(0)$, a search equilibrium consists of price functions p_r and p_d, seller expected utility functions V_r and V_d, population functions h_r and h_d, and buyer strategy function μ at each point in time such that:

1. Seller utility correctly anticipates market conditions (Eqs. 3 and 5).

2. Prices are optimally set for their respective targets (Eqs. 4 and 6).

3. Seller populations obey the law of motion (Eqs. 7 and 8).

4. Buyers use optimal pricing strategies: $\mu = 0$ if $\Pi < 0$ and $\mu = 1$ if $\Pi > 0$.

The fourth requirement ensures that buyers offer the price with the highest expected payoff. If $\Pi = 0$, then any $\mu \in [0, 1]$ is admissible, though the other equilibrium requirements will pin down its value. If offering either price is equally profitable, $\Pi = 0$ rearranges to imply a specific relationship between the two prices:

$$p_r = \phi p_d + \frac{y(1-\phi)}{\rho} \text{ if } \mu \in (0, 1). \tag{9}$$

3 Equilibrium Characterization

We now solve these equilibrium conditions, first for steady state and then for the dynamic transition. When buyers offer both prices at the same time ($\mu \in (0, 1)$), we refer to this as a *dispersed equilibrium*. If a degenerate price distribution arises, we label this a *relaxed equilibrium* or a *desperate equilibrium*, depending on the price offered.

3.1 Steady State Equilibrium

To solve for the steady state equilibrium, we find initial conditions $h_r(0)$ and $h_d(0)$ such that $\dot{h}_r = \dot{h}_d = 0$ for all t. Two possibilities exist, depending on parameter values. In the *relaxed steady state* equilibrium, only the relaxed sellers are targeted because there are too few desperate sellers to be worth offering p_d. In the *dispersed steady state* equilibrium, both prices are offered.[8]

The solutions are reported below in Table 1. For notational brevity, we define:

$$\beta \equiv \frac{\rho}{\rho + \lambda} \cdot \frac{c}{y - x}. \tag{10}$$

This measure of discounted delay costs determines whether price dispersion occurs in steady state. The dispersed steady state occurs when $\beta\delta > \eta$, so that $\mu \in (0, 1)$. Moreover, the relaxed steady state occurs if $\beta\delta \leq \eta$: buyers do not target desperate sellers if there are relatively few of them entering, they are fairly patient, or their cost of delay is small.[9]

[8] A third scenario where only desperate prices are offered is not possible in steady state, because relaxed sellers would never exit, causing h_r to grow without bound.

[9] This can be seen by inserting the equilibrium prices and populations into Eq. 2.

Table 1: Steady State Solution

	Dispersed	Relaxed
	occurs when	
	$\beta\delta > \eta$	$\beta\delta \leq \eta$
p_r	$\frac{x}{\rho}$	
p_d	$\frac{x-c}{\rho} + \frac{\eta\lambda(y-x)}{\delta\rho^2}$	$\frac{x-\beta(y-x)}{\rho}$
h_r	$\frac{\beta\delta(\rho+\lambda)}{\rho\lambda} - \frac{\eta}{\rho}$	$\frac{\eta}{\lambda}$
h_d	$\frac{\delta}{\lambda}$	
μ	$1 - \frac{\rho\eta}{(\rho+\lambda)(\beta\delta-\eta)+\rho\eta}$	0

3.2 Characterization of Dynamic Paths

When the initial population of relaxed sellers $h_r(0)$ and/or desperate sellers $h_d(0)$ are not at their steady state levels, we can derive their unique transition paths. It is important to note that this is a deterministic transition, with a future path that is commonly known by all buyers and sellers. Absent any frictions, the potential for arbitrage would force an immediate transition to the long run outcome. Despite knowing how prices will adjust over time, buyers and sellers in our model only occasionally have opportunities to interact; thus, their populations only adjust slowly towards the long run.

The dynamic path can be characterized as passing through phases. When a shock to the economy causes an imbalance in populations of seller types, buyers will typically target the relatively plentiful group, generating either a *desperate* or *relaxed* phase. In a *dispersed* phase, buyers target both types of sellers.

Lemma 1 and the accompanying Table 2 establish the solution for each phase. We

Table 2: Dynamic Paths

	Desperate Phase	Dispersed Phase	Relaxed Phase
	occurs when		
	$\Pi(t) > 0$	$\Pi(t) = 0$ and $\mu(t) \in (0,1)$	$\Pi(t) < 0$
$p_r(t)$	$\frac{x}{\rho}$		
$p_d(t)$	$\frac{x-c}{\rho} + a_d e^{\rho t}$	$\frac{x}{\rho} - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$	$\frac{x-\beta(y-x)}{\rho} + a_r e^{t(\rho+\lambda)}$
$h_r(t)$	$\frac{\eta}{\lambda} + q_r + \eta t$	$\Delta(h_d(t))$	$\frac{\eta}{\lambda} + q_r e^{-\lambda t}$
$h_d(t)$	$\frac{\delta}{\lambda} + q_d e^{-\lambda t}$		
$\mu(t)$	1	$1 - \frac{\eta - h_r'(t)}{h_r(t)}$	0

let q_d denote the difference between the initial population of desperate sellers and $\frac{\delta}{\lambda}$ (its steady state level), while q_r denotes the difference between the initial population of relaxed sellers and $\frac{\eta}{\lambda}$ (its level in the relaxed steady state). We also define $\Delta(h_d)$ as the function:

$$\Delta(h_d) \equiv \beta h_d + \frac{\beta\delta - \eta}{2\lambda} \int_0^1 \frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1 - s + \frac{\delta}{\lambda h_d} s\right)^{\frac{1}{2}}} ds, \tag{11}$$

which specifies the relaxed population needed to sustain a dispersed equilibrium for a given desperate population.

Lemma 1. *Given initial seller populations $h_d(0) = \frac{\delta}{\lambda} + q_d$ and $h_r(0) = \frac{\eta}{\lambda} + q_r$, a dynamic equilibrium path is characterized by one of the three phases listed in Table 2, for some constant a_r or a_d.*

In any phase, the relaxed price is constant at $\frac{x}{\rho}$. Intuitively, the relaxed sellers have no reason to accept a lower price, since this exactly replaces their current utility;

and buyers have no reason to make a larger take-it-or-leave it offer, since everyone is willing to accept that price.

Turning to desperate population dynamics, note that the same transition path applies in any phase. This holds because desperate sellers accept every offer that is made in equilibrium and thus exit the market at a constant rate λ. Also, the population monotonically approaches its steady state level over time, or stays at that level for all t if $q_d = 0$.

Relaxed population dynamics are similar in a relaxed phase, monotonically approaching the relaxed steady state level over time. However, in a desperate phase, relaxed sellers reject all offers and thus their population increases over time. This cannot increase indefinitely, but rather will eventually transition into a dispersed phase, where buyers resume offers to relaxed sellers.

The dispersed path requires that buyers are indifferent between making low or high price offers. This can only happen if the proportion of relaxed and desperate sellers is right: each $h_d(t)$ requires a particular $h_r(t)$, as indicated by $h_r(t) = \Delta(h_d(t))$. Indeed, the set of (h_d, h_r) pairs that can be part of a dispersed transition path form a one-to-one mapping, with h_r increasing in h_d.[10]

The solution for the desperate price is unique up to a constant — a_d for the desperate phase or a_r for the relaxed phase — which are pinned down in the next subsection by transitions between phases. Here we note that the desperate price is increasing over time if and only if this constant is positive.

In the next subsections, we consider how these phases interact to form a full dynamic path. This behavior differs depending on whether the unique steady state has degenerate or dispersed prices.

[10]Since $h_d(t) \to \frac{\delta}{\lambda}$ as $t \to \infty$, the integral in $\Delta(h_d)$ will approach $\frac{2\lambda}{\rho}$, and thus $h_r(t)$ approaches its dispersed steady state value.

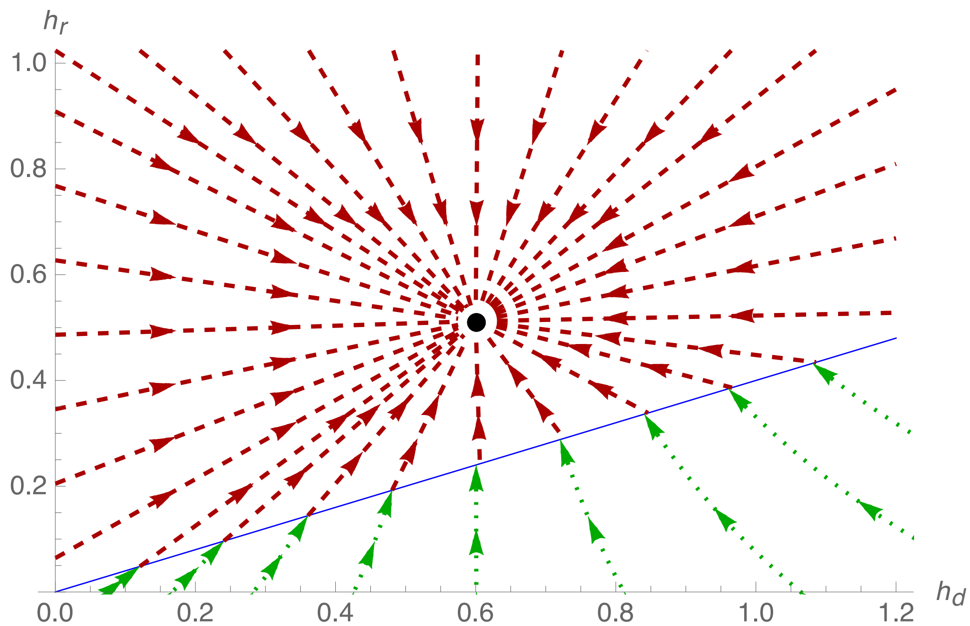

Figure 1: **Degenerate Phase Portrait** indicating the path of seller populations from any initial state, with dashed arrows in a relaxed phase and dotted arrows in a desperate phase. The solid line indicates the boundary between these phases. ($y = 5.5$, $x = 5$, $c = 5$, $\eta = 1.28$, $\delta = 1.5$, $\rho = 0.1$, and $\lambda = 2.5$)

3.3 Transition toward a Relaxed Steady State

First, suppose that parameters are such that only the relaxed price will be offered in steady state. We consider the dynamic transition starting from an arbitrary initial population. Figure 1 illustrates these transition paths in a phase portrait depicting populations of both seller types, with the steady state populations indicated by the dot. When the path is dashed, it indicates that buyers only offer relaxed prices; when it is dotted, buyers only offer desperate prices. Buyers are only indifferent between offering the two prices along the solid line.

In the dashed region, buyers only offer the relaxed price due to the abundance of relaxed sellers. For instance, starting from $h_d = 0$ and $h_r = 0.2$, both populations are below their steady state levels (0.6 and 0.5, respectively); therefore, new entrants (δ or η) outpace exiting sellers (λh_d or λh_r). Every seller accepts every offer, so both populations rise on a straight trajectory toward steady state. We note that prices p_r and p_d stay at their steady state values throughout this region.

In the dotted region, relaxed sellers are scarce, so buyers find it optimal to initially target desperate sellers. The population of desperate sellers may rise or fall, depending on whether they are below or above the steady state, respectively; but the population of relaxed sellers steadily climbs as they enter the market but accept no offers. This ensures that eventually buyers will shift to target relaxed sellers. For instance, starting from $h_d = 0.43$ and $h_r = 0$, both populations follow the desperate path until $h_d = 0.47$ and $h_r = 0.2$, after which they continue on a relaxed path.

Significantly, there cannot be a dispersed phase of the dynamic transition where both prices are offered. In the interior of either region, buyers would be strictly worse off if they targeted the other seller. On the boundary between these regions (where $h_r = \beta h_d$), offering either price is equally profitable for an instant. Yet a dispersed equilibrium would have relaxed sellers still rejecting some offers, so h_r would grow faster than h_d, making p_d offers unprofitable thereafter.

Perhaps the most interesting behavior in dynamic transitions to a relaxed steady state is the discrete shift from desperate to relaxed phases. This would translate into a realized transaction price (p_d) that starts low and gradually rises over time, hitting its steady state value at the switch to the relaxed phase. However, at that instant, the realized transaction price jumps, as buyers switch to only offer p_r. This happens either when relaxed sellers start well below their steady state level, or when desperate sellers start well above theirs.

Proposition 1 establishes that this behavior holds generally.

Proposition 1. *Suppose that $\beta\delta \leq \eta$, so that the relaxed degenerate steady state exists. Given initial populations $h_d(0)$ and $h_r(0)$:*

- ***Case 1:*** *If $h_r(0) \geq \beta h_d(0)$, then for all t the dynamic transition must follow the relaxed phase with $a_r = 0$ until reaching steady state.*

- ***Case 2:*** *If $h_r(0) < \beta h_d(0)$, then for $t < T$ the dynamic transition must follow the desperate phase with $a_d = \frac{\lambda c}{\rho(\rho+\lambda)} e^{-\rho T}$, where T satisfies $h_r(T) = \beta h_d(T)$. For $t \geq T$, it must follow the relaxed phase with $a_r = 0$ until reaching steady state.*

3.4 Transition toward a Dispersed Steady State

Next, suppose that parameters would lead to both prices being offered in the unique steady state. This produces richer behavior in the transition paths. For a narrow set of initial seller populations (of measure zero), the transition always remains in a dispersed phase. Generically, however, the path will start on a relaxed or desperate phase, then eventually reach the dispersed phase (which forms the boundary between the two phases as illustrated in Panel B of Figure 2). It is even possible to begin in a relaxed phase, transition to a desperate phase, then conclude in a dispersed phase — which we refer to as a *bifurcated* path.

Thus, the space of possible initial conditions (for h_d and h_r) is partitioned into four regions, illustrated in Panel A of Figure 2, each yielding a unique path to steady state for each initial condition. Specific transition paths are exemplified in Panel B of Figure 2. These regions and phase portrait are representative of the general behavior, which we formally show in Proposition 2 at the end of this section.

In the dispersed region (the solid line in Figure 2.A and B), both prices are offered throughout the dynamic transition, and seller populations approach steady state by following the solid line. For example, at $h_d = 2$ and $h_r = 4$, the dispersed phase will gradually approach the steady state of $h_d = 1.6$ and $h_r = 3.2$. In the space of all pos-

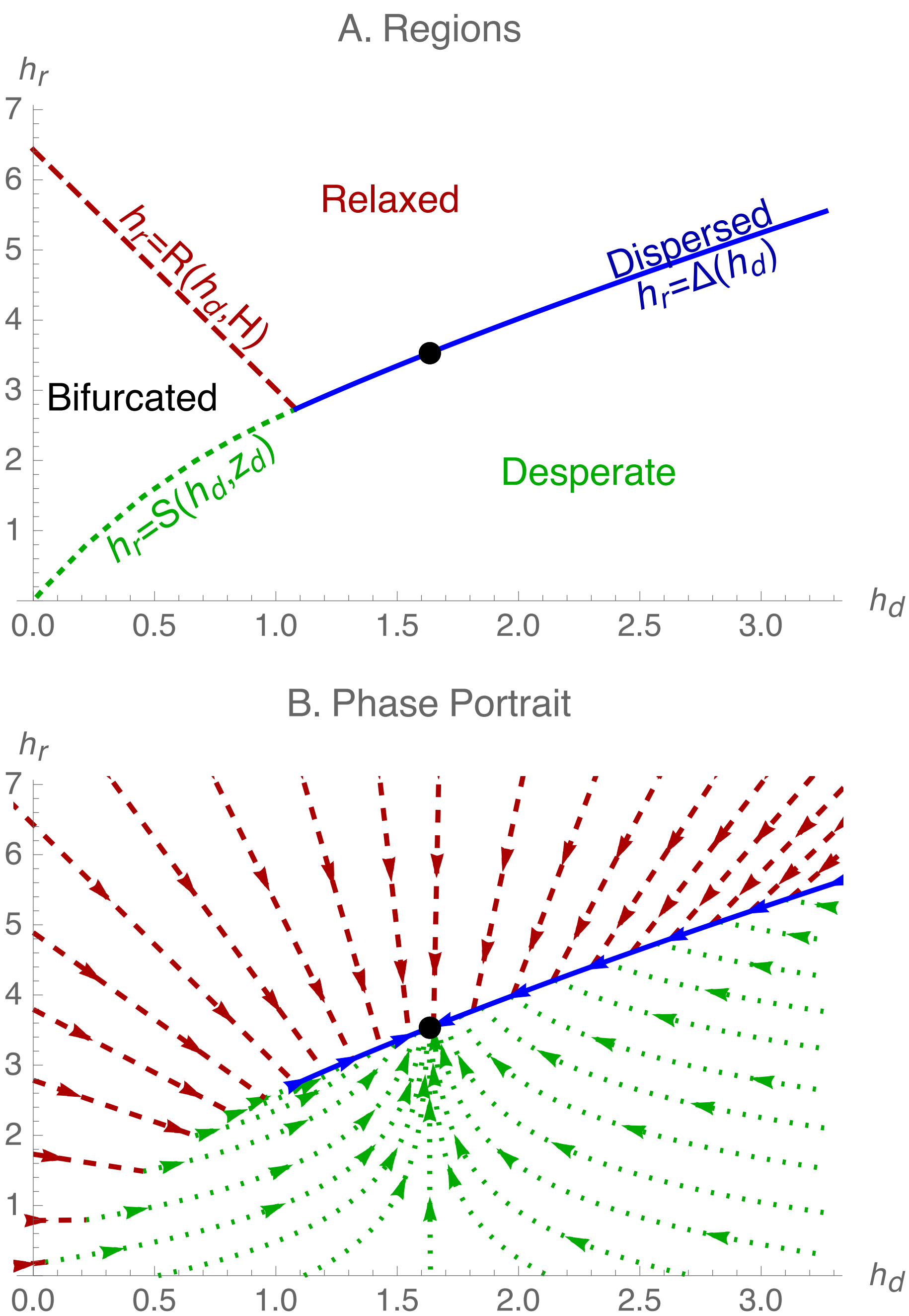

Figure 2: **Transitions to a Dispersed Steady State:** Panel A reports the initial phase of the transition path for any initial state (h_d, h_r). Panel B indicates the path of seller populations from any initial state. The solid dot marks the dispersed steady state populations. ($y = 5.5$, $x = 5$, $c = 5$, $\eta = 1.28$, $\delta = 2.45$, $\rho = 0.1$, and $\lambda = 1.5$)

sible initial population levels, this dispersed region has measure zero, indicating that price dispersion almost never continues immediately following a random population shock. Even so, the other regions eventually feed into this dispersed path; thus, price dispersion is later observed in some part of almost every transition path. While not depicted in the figure, the desperate price p_d is still adjusting along this path, rising if h_d is below its steady state level of $\frac{\delta}{\lambda}$, and falling otherwise. Meanwhile, the fraction of desperate offers μ moves in the opposite direction.

In the relaxed region (above the dashed and solid lines in Figure 2.A), relaxed sellers are more plentiful, so buyers exclusively offer p_r initially. This eventually draws down the relaxed population sufficiently to reach the dispersed path (dashed lines reaching the solid line in Figure 2.B), at which point it becomes profitable to offer both prices. For example, starting from $h_d = 2.3$ and $h_r = 7$, the relaxed phase will reduce both populations to $h_d = 2$ and $h_r = 4$, whereupon the preceding dispersed phase begins. Indeed, it is convenient to describe a relaxed path relative to the $\hat{h}_d$ where it intersects the dispersed path. Any pair $(h_d, R(h_d, \hat{h}_d))$ lies on a relaxed path leading to $\left(\hat{h}_d, \Delta(\hat{h}_d)\right)$, where:

$$R(h_d, \hat{h}_d) \equiv \frac{(h_d - \hat{h}_d)\eta + (\delta - \lambda h_d)\Delta(\hat{h}_d)}{\delta - \lambda \hat{h}_d}, \tag{12}$$

as derived in the proof of Proposition 2. We also note that $a_r < 0$, meaning the price desperate sellers are willing to accept falls until reaching the dispersed path.

In the desperate region (below the dotted and solid lines in Figure 2.A), relaxed sellers are relatively scarce. Thus, buyers find it profitable to exclusively offer p_d initially. This ensures that the population of relaxed sellers steadily grows until eventually reaching the dispersed path (dotted lines reaching the solid line in Figure 2.B). For example, starting from $h_d = 3.3$ and $h_r = 2.6$, the desperate phase will decrease the desperate population to $h_d = 2$ while building the relaxed population to $h_r = 4$, whereupon the preceding dispersed phase begins. We similarly depict any pair $(h_d, S(h_d, \hat{h}_d))$ as part of a desperate path leading to $\left(\hat{h}_d, \Delta(\hat{h}_d)\right)$, where:

$$S(h_d, \hat{h}_d) \equiv \Delta(\hat{h}_d) - \frac{\eta}{\lambda} \ln \frac{\delta - \lambda h_d}{\delta - \lambda \hat{h}_d}. \tag{13}$$

In this region, $a_d > 0$, which implies that desperate sellers insist on higher prices over time until reaching the dispersed path.

Finally, the bifurcated region (between the dashed and dotted lines in Figure 2.A) has very few desperate sellers and a moderate number of relaxed sellers. For example, consider $h_d = 0$ and $h_r = 2.8$. This leads buyers to initially target the relaxed sellers, but once that population is drawn down ($h_d = 0.65$ and $h_r = 2$), the desperate sellers become a strictly more attractive target (dashed lines lead to dotted lines in Figure 2.B). This leads to a surprising discrete drop in the realized price offers, followed by a steady recovery in a desperate phase until reaching the dispersed phase (at $h_d = 1.4$ and $h_r = 3.3$).

Note that the dispersed path in Figure 2.A does not extend below $h_d = 1.1$ (the intersection of the regions); at lower levels, the proposed μ would be strictly greater than one and hence would not be a mixed strategy. Indeed, this is why the relaxed phase transitions to a desperate phase in the bifurcated region rather than entering a dispersed phase. To formally define the point on the dispersed path at which $\mu = 1$, let $H > 0$ solve:

$$\Delta(H) = \frac{\eta(y - x) + \rho c H}{(y - x)(\delta + H(\rho - \lambda))} H. \tag{14}$$

We are then able to characterize the unique equilibrium path for any initial condition as follows. In the proposition, $\hat{h}_d$ indicates where the relaxed or desperate path intersects the dispersed path. In the bifurcated case, $\tilde{h}_d$ indicates where the relaxed path hits $\Pi = 0$ and thus transitions to a desperate path.

Proposition 2. *Suppose that $\beta > \eta/\delta$, so that the dispersed steady state exists, and that initial populations are $h_d(0) = \frac{\delta}{\lambda} + q_d$ and $h_r(0) = \frac{\eta}{\lambda} + q_r$.*

- ***Case 1 (Dispersed):*** *If $h_r(0) = \Delta(h_d(0))$ and $h_d(0) \geq H$, then the transition follows the dispersed path until reaching steady state.*

- ***Case 2 (Relaxed):*** *Let $\hat{h}_d$ be the solution to $h_r(0) = R\left(h_d(0), \hat{h}_d\right)$. If $h_r(0) > \Delta(h_d(0))$ and $\hat{h}_d \geq H$, then the transition follows the relaxed phase while $t < T_r \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \hat{h}_d - \delta}$, where $a_r = \frac{(y-x)(\beta \hat{h}_d - \Delta(\hat{h}_d))}{\rho \hat{h}_d} e^{-(\rho+\lambda)T_r}$. For $t \geq T_r$, it follows the dispersed path until reaching steady state.*

- ***Case 3 (Desperate):*** *Let $\hat{h}_d \geq H$ be the solution to $h_r(0) = S\left(h_d(0), \hat{h}_d\right)$. If $h_r(0) < \Delta(h_d(0))$ and $h_r(0) < \frac{c-\rho a_d}{y-x} h_d(0)$, then the transition must follow the desperate phase while $t < T_d \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \hat{h}_d - \delta}$ with $a_d = \frac{c\hat{h}_d - (y-x)\Delta(\hat{h}_d)}{\rho \hat{h}_d} e^{-\rho T_d}$. For $t \geq T_d$, it must follow the dispersed path until reaching steady state.*

- ***Case 4 (Bifurcated):*** *Otherwise, let $\hat{h}_d \geq H$ and $\tilde{h}_d$ be the solutions to $S\left(\tilde{h}_d, \hat{h}_d\right) = \frac{\eta}{\lambda} + \frac{(\lambda \tilde{h}_d - \delta) q_r}{\lambda q_d}$ and $S(\tilde{h}_d, \hat{h}_d) = \frac{c-\rho a_d}{y-x} \tilde{h}_d$. Then the transition must follow the relaxed phase while $t < T_b \equiv \frac{1}{\lambda} \ln \frac{\lambda q_d}{\lambda \tilde{h}_d - \delta}$ with $a_r = a_d e^{-\lambda T_b} - \frac{\lambda c}{\rho(\rho+\lambda)} e^{-(\rho+\lambda)T_b}$. This is followed by a desperate phase while $t < T_d \equiv T_b + \frac{1}{\lambda} \ln \frac{\lambda \tilde{h}_d - \delta}{\lambda \hat{h}_d - \delta}$ with $a_d = \frac{c\hat{h}_d - (y-x)\Delta(\hat{h}_d)}{\rho \hat{h}_d} e^{-\rho T_d}$. For $t \geq T_d$, it must follow the dispersed path until reaching steady state.*

4 Experiments

Our purpose is to shed light on price dynamics of durable assets during economic cycles. The macro finance literature documents the counter-cyclicality of fire sales—

that is, during recessions, assets are more frequently sold below their fundamental value and during expansions, prices recover to their pre-downturn levels. A downturn can be simulated in our model by a temporary increase in the number of desperate sellers with high liquidity needs. Their relative abundance in the market would push prices down, and then lead its way towards recovery as desperate sellers make more frequent sales and exit the market.

In this section, we present experiments that showcase our model economy's response to various unanticipated shocks: temporary or permanent. For temporary shocks (Experiments 1 and 2), we consider a sudden infusion of sellers of one type or the other; that is, we start from an $h_r(0)$ or $h_d(0)$ away from steady state and follow the market behavior thereafter. This does not affect the steady state, but the frictions inherent in random matching prevent instantaneous adjustment. For permanent shocks (Experiments 3 and 4), we begin with both initial seller populations at their steady state levels, then consider what would occur if one of the parameters were to unanticipatedly and permanently change.

4.1 Experiment 1: Additional Desperate Sellers

Suppose that the market experiences a sudden influx of additional desperate sellers without affecting the number of relaxed sellers. For instance, banks may suddenly need to sell a large stock of foreclosed properties to improve their balance sheets. In Figure 3, we doubling the number of desperate sellers at $t = 0$.

This experiment can be seen as jumping to the right edge of the phase portrait in Figure 2.B. Thus, the transition begins in a desperate phase, then eventually reaches the dispersed phase. In Panel A of Figure 3, the adjusting population levels are shown as a function of time since the shock. Since desperate sellers are relatively abundant

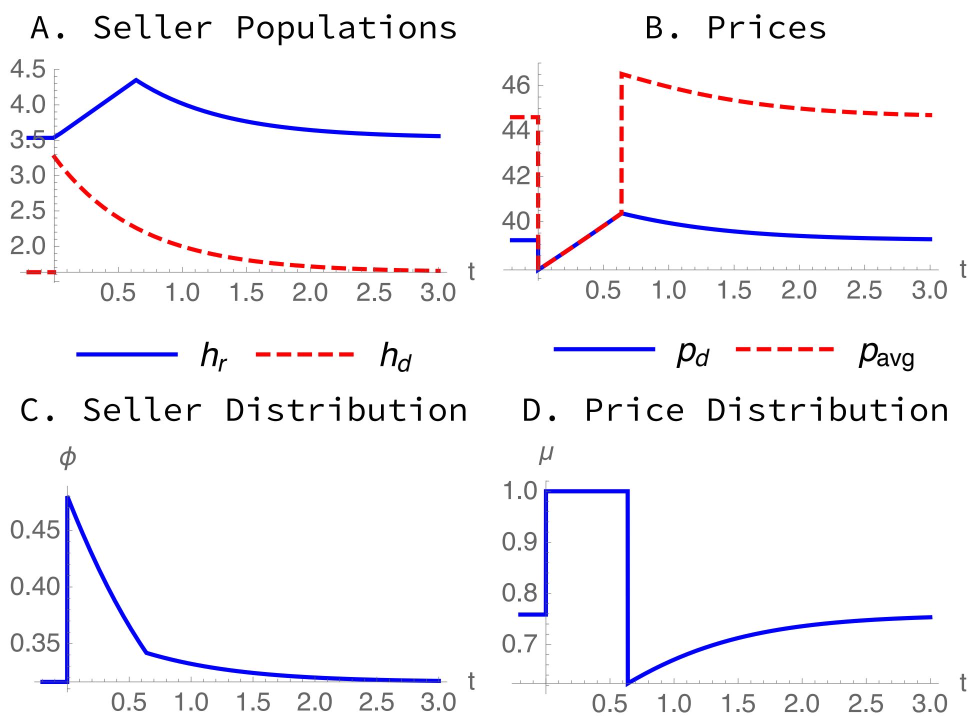

Figure 3: Dynamic response from doubling the number of desperate sellers at time $t = 0$

after the shock (ϕ jumps upward in Panel C), buyers exclusively target them initially (μ jumps to 1 from its steady-state level of 0.75 in Panel D). During this desperate phase, relaxed sellers continue to enter the market but reject all offers; meanwhile, desperate sellers accept offers faster than they are replaced by new entrants. Together, this reduces ϕ at a fast pace until a critical time ($t = 0.6$ on the figure) at which some buyers resume targeting the relaxed sellers (μ drops to 0.62).

Beyond what was evident in the phase portrait, Figure 3 also indicates how the price distribution changes through this transition. The solid line in Panel B depicts a non-monotonic transition in the desperate price, which shadows the fluctuation in future alternatives for the desperate sellers. Following the shock, this price discretely drops 2% because buyers no longer offer the relaxed price. However, desperate sellers anticipate the return of some full price offers to the market, so they gradually insist on higher prices (rising 4% over the desperate phase) as the critical time approaches. Finally, the dispersed phase begins with more relaxed offers than in steady state; as the proportion of desperate offers gradually climbs from $\mu = 0.62$ to 0.75, seen in Panel D, desperate sellers are willing to accept lower price offers.

The non-monotonic price transition is even more dramatic if measured in terms of realized transaction prices. The dashed line in Panel B indicates the average price at which the asset is sold. During the desperate phase, we see a collapse in asset prices, since all relaxed price offers disappear. When the dispersed phase begins, this jump is more than reversed by the reintroduction of relaxed offers.

4.2 Experiment 2: No Desperate Sellers

Next, consider a sudden drop in the number of desperate sellers. This might occur if a one-time intervention refinances underwater homes, allowing the owners to remain in their homes. Figure 4 depicts the response in the extreme case where all desperate sellers suddenly exit the market.

This experiment would appear as jumping to the left edge of the phase portrait in Figure 2.B, passing through relaxed, then desperate, then dispersed phases. Panel A of Figure 4 charts these phases in the population adjustment, with the relaxed phase ending at $t = 0.45$ and the desperate phase ending at $t = 1.15$. Immediately following the shock, there are no desperate sellers (ϕ drops to 0 in Panel C), so buyers stop making desperate offers entirely (μ drops to 0 in Panel D). Since the relaxed sellers now accept every offer, they exit at a faster rate and their population declines; meanwhile, newly-entering desperate sellers gradually build their population.

By $t = 0.45$, the ratio of desperate sellers ϕ has nearly recovered, and entices buyers to target them. Indeed, they are indifferent in only that exact moment, be-

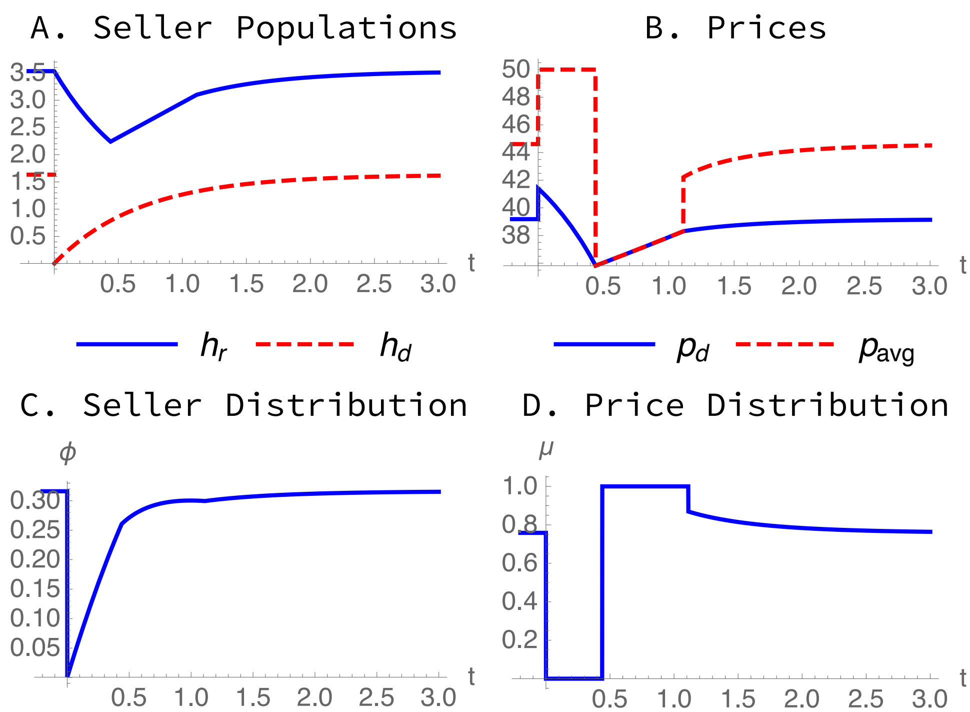

Figure 4: Dynamic response from eliminating all desperate sellers at time $t = 0$

cause the proportion of desperate sellers rises further thereafter. Thus, the buyers completely reverse their strategy to exclusively target the desperate (μ jumps to 1 in Panel D). This prevents any relaxed sellers from exiting the market, and thus their population begins to recover. By $t = 1.15$, buyers are willing to make both types of offers (dropping to $\mu = 0.85$), and gradually reduce their desperate offers thereafter to approach the steady state level of $\mu = 0.75$.

In Panel B, the reservation price of desperate sellers initially jumps upward since all buyers are offering the relaxed price. However, it then falls as the desperate phase approaches ($t = 0.45$), then rises thereafter in anticipation of favorable offers in the dispersed phase.

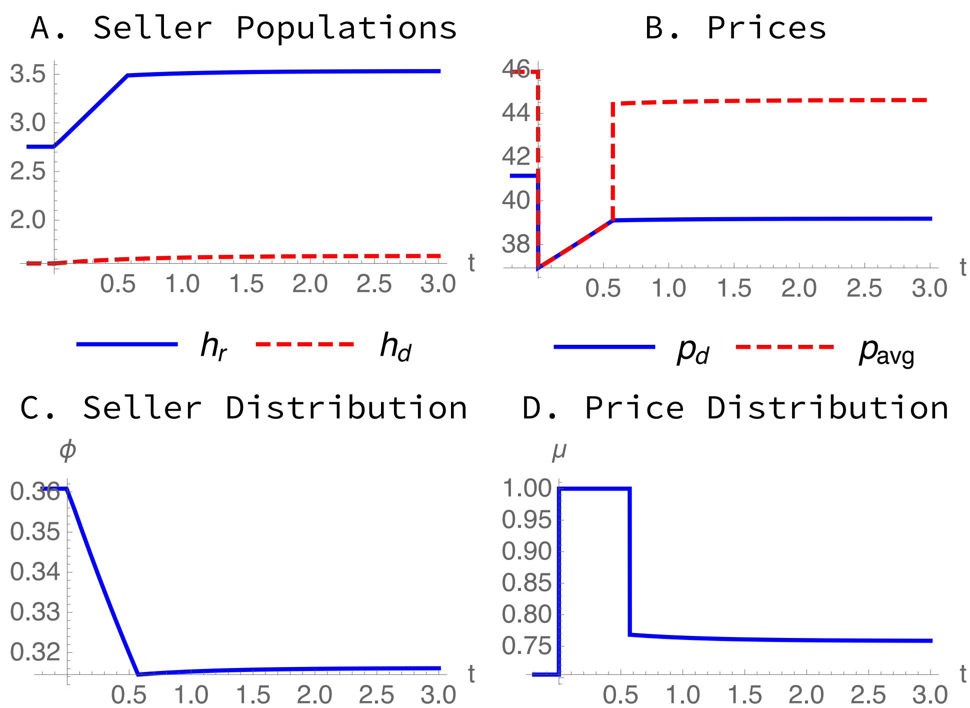

Figure 5: Dynamic response from a permanent 5% decrease in the matching rate λ, starting at time $t = 0$

4.3 Experiment 3: Higher Search Frictions

Now consider a permanent decrease in the rate at which sellers encounter buyers. This could reflect having fewer buyers available in the market, or perhaps that each match takes longer due to more paperwork or inspections. Figure 5 presents the dynamic response from a 5% decline in λ.[11]

The new steady state has more sellers of both types, but relatively more relaxed sellers. If the new steady state is depicted at the center of the phase portrait in Figure 2.B, the old steady state lies in the lower left quadrant. This leads to an initial desperate phase, during which the population of relaxed sellers quickly climbs as none accept offers (Figure 5.A). The desperate population also climbs due to the slower rate of offers, though at a much slower pace, causing ϕ to plummet (Panel C).

[11] An increase in δ produces nearly identical response.

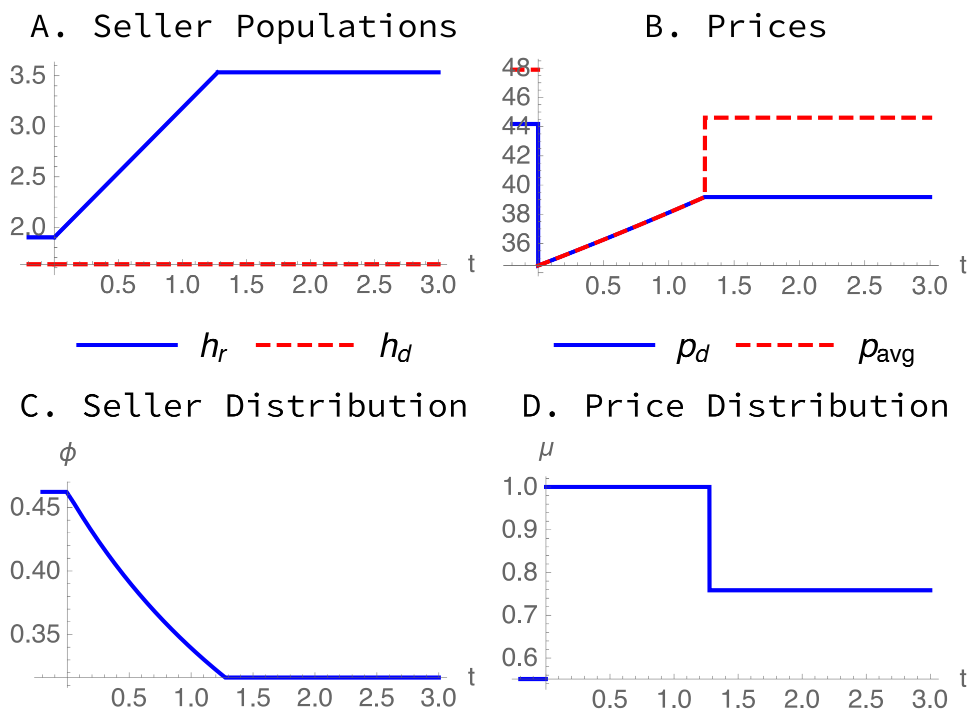

Figure 6: Dynamic response from a permanent 10% increase in the cost to desperate sellers, c, starting at time $t = 0$

Thus, most of the adjustment occurs before reaching the critical time $t = 0.6$ where a dispersed phase begins. Thereafter, both populations continue to slowly grow, but the desperate grow slightly faster causing ϕ to rebound slightly. The desperate price drops downward at the time of the shock then slowly recovers (Panel B).

4.4 Experiment 4: Higher Desperation Cost

Lastly, consider a higher cost of waiting, which widens the gap between the asset value to relaxed versus desperate sellers, perhaps reflecting higher personal costs of bankruptcy. Figure 6 depicts the impact of a 10% increase in this cost.[12]

The steady state number of desperate sellers is unaffected. In Figure 2.B, the old

[12]An identical response is produced from a decrease in η, and highly similar behavior comes from an increase in x or decrease in y.

steady state would lie directly below the new steady state in the center of the graph, and thus the transition fully occurs within the desperate phase. This can be seen in Figure 6.A, since h_r reaches its steady state value at $t = 1.25$, rather than merely approaching it in the limit. This population shift occurs because desperate sellers are abundant, relative to the new steady state (Panel C), so buyers exclusively target them initially (Panel D). At the critical time, however, the seller populations reach the required steady state, in both their levels and proportion simultaneously. Thus, further adjustment is unnecessary. Prices first drop below the new steady state level, then steadily recover (Panel B).

5 Extensions

5.1 Entry Model

In our baseline model, buyers optimally choose which sellers to target, but only in instantaneous decisions. In this extension, we allow for inter-temporal planning by buyers, disciplining it with competition among buyers. That is, buyers will correctly anticipate the equilibrium path of seller populations and prices, but free entry and exit by the buyers will ensure that expected profits are zero. Relative to our baseline model, this extension endogenizes λ; yet the equilibrium behavior is largely unchanged.

Assume that the rate of meetings in the market follow a Cobb-Douglas matching function with equal weights on buyers and sellers: $m \equiv \psi g^{\frac{1}{2}} h^{\frac{1}{2}}$. Thus, a seller encounters buyers at rate $\lambda \equiv \frac{m}{h}$, while buyers encounter sellers at rate $\alpha \equiv \frac{m}{g}$.

Next, assume that buyers in the market pay a search cost k per period. Let B_r

denote the present value of expected profit for a buyer that always offers p_r:

$$\rho B_r = -k + \dot{B}_r + \alpha \left(\frac{y}{\rho} - p_r - B_r \right). \tag{15}$$

Here, time is a state variable because the rate of meeting could change over time. In the last term, meetings occur at rate α, and since the proposed price is always accepted, the buyer will stop searching ($-B_r$) and realize profit of selling ($\frac{y}{\rho} - p_r$).

If the buyer always offers p_d, the present value of expected profit B_d is:

$$\rho B_d = -k + \dot{B}_d + \alpha\phi \left(\frac{y}{\rho} - p_d - B_d \right). \tag{16}$$

As before, ϕ indicates the probability that an offer is rejected (by relaxed sellers).

Since buyers can freely enter either market, we require that $B_r \leq 0$ and $B_d \leq 0$. On the other hand, free exit ensures that no buyer must endure losses. This requirement can be expressed as $\mu B_d = 0$ and $(1-\mu)B_r = 0$. We note that relative to our baseline model, instantaneous targeting decision are unaltered (*i.e.* $\Pi > 0$ iff $B_d > B_r$).

The combination of free entry and the matching function allow us to solve for the endogenous λ as:

$$\lambda = \begin{cases} \frac{\psi^2(y-x)}{\rho k} & \text{if } \mu < 1 \\ \frac{\phi\psi^2(y-\rho p_d)}{\rho k} & \text{if } \mu = 1. \end{cases} \tag{17}$$

In a relaxed or dispersed phase ($\mu < 1$), the solution to λ is a constant. The realized profit when a buyer trades with a relaxed seller is constant over time; thus, even when the populations of sellers change, buyers proportionally enter or exit to compete away excess profits. Thus, relaxed and dispersed phases proceed exactly as in the baseline model (under the right ψ and k). However, when all buyers target the desperate sellers ($\mu = 1$), the realized profit falls as price p_d increases over the desperate phase, causing λ to fall over time.

This is also why a full analytic characterization is no longer possible when buyer entry is endogenous. While the relaxed and dispersed phase proceed as before, the desperate population law of motion cannot be analytically solved in the desperate phase. While h_r and p_d can be solved directly and substituted in, the following differential equation in h_d must be numerically solved:

$$\dot{h}_d = \delta - \frac{\psi^2(y - \rho p_d)}{\rho k} \cdot \frac{h_d^2}{h_d + h_r}. \tag{18}$$

The numerical solution bears strong resemblance to the baseline model.[13] In Figure 7, we present a phase portrait for the buyer entry model, using the same parameter values as in Figure 2 and setting $\psi = 1$ and $k = 3.33$ so that the equilibrium $\lambda = 1.5$ in the relaxed or dispersed phases as before.

The relaxed and dispersed phases are literally unchanged from Figure 2.B. A desperate path still occurs in the same region, but has greater curvature than when λ is exogenous. Indeed, starting with many desperate and few relaxed sellers, enough extra buyers will enter the market to quickly draw down the desperate population, which drops even below the steady state. As this happens, the desperate price climbs until buyers begin leaving the market. This slows down the arrival rate λ and allows h_d to grow toward steady state. The population of relaxed sellers steadily grows as before, since they accept no offers.

[13]Indeed, near the dispersed path, we can prove that the behavior is identical to the baseline model.

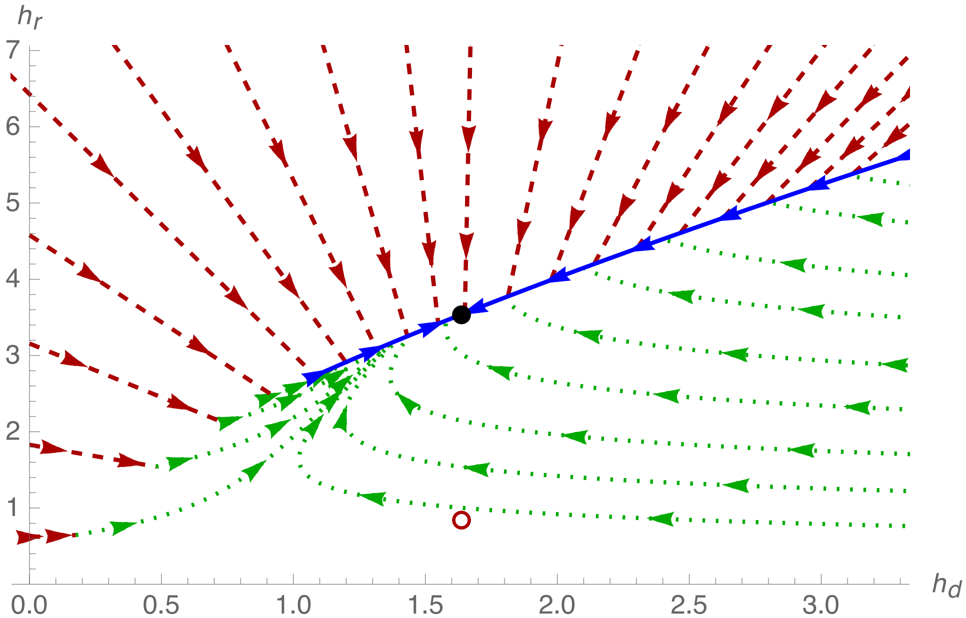

Figure 7: **Phase Portrait with endogenous buyer entry**. The solid dot indicates the dispersed steady state, while the solid line denotes the dispersed phase of the transition path. Dashed and dotted lines indicate the relaxed and desperate phases, respectively. The open dot indicates the populations if the relaxed steady state had existed.

5.2 Urgency Model

Our baseline model captures heterogeneity in the marketplace by exogenously assuming that some sellers enter the market with greater urgency to sell. An alternative approach would allow relaxed sellers to become desperate at some point in their search. For instance, a major life event like a wedding or job change may require liquidation of financial or real estate assets. We assume this transition randomly arrives at Poisson rate τ. We refer to this as the Urgency model.

This has three effects on our model setup. First, relaxed seller population dynamics now include the exit of those who become desperate at rate τh_r:

$$\dot{h}_r = \eta - \lambda(1-\mu)h_r - \tau h_r. \tag{19}$$

Table 3: Urgency Model: Steady State Solution

	Desperate	Dispersed	Relaxed
	occurs when		
	$\gamma \geq \frac{c}{\rho+\tau}$	$\frac{c}{\rho+\tau} > \gamma > \frac{c}{\rho+\tau+\lambda}$	$\frac{c}{\rho+\tau+\lambda} \geq \gamma$
p_r	$\frac{x}{\rho} - \frac{\tau c}{\rho(\rho+\tau)}$	$\frac{x-\tau\gamma}{\rho}$	$\frac{x}{\rho} - \frac{\tau c}{\rho(\lambda+\rho+\tau)}$
p_d	$\frac{x-c}{\rho}$	$\frac{x-(\rho+\tau)\gamma}{\rho}$	$\frac{x}{\rho} - \frac{c(\rho+\tau)}{\rho(\lambda+\rho+\tau)}$
h_r	$\frac{\eta}{\tau}$	$\frac{\eta\gamma}{c-\rho\gamma}$	$\frac{\eta}{\lambda+\tau}$
h_d	$\frac{\delta+\eta}{\lambda}$	$\frac{\delta c+(\tau\eta-\rho)\gamma}{\lambda(c-\rho\gamma)}$	$\frac{\delta+\eta}{\lambda} - \frac{\eta}{\lambda+\tau}$
μ	1	$1 + \frac{\rho+\tau}{\lambda} - \frac{c}{\lambda\gamma}$	0

Second, the desperate population is augmented by this flow of relaxed sellers.

$$\dot{h}_d = \delta + \tau h_r - \lambda h_d. \tag{20}$$

Finally, relaxed sellers anticipate this random change in state which occurs at rate τ and moves them from V_r to V_d:

$$\rho V_r = x + \dot{V}_r + \lambda(1-\mu)\left(p_r - V_r\right) + \tau(V_d - V_r). \tag{21}$$

The urgency model offers three potential steady states. Under the right parameters, it is possible to sustain a desperate steady state, in which buyers only offer the desperate price. Relaxed sellers never exit the market due to selling, but eventually become desperate and then make a sale. The three solutions are reported in Table 3, and they are mutually exclusive, resulting in a unique steady state.

For notational simplicity, we introduce the parameter γ, which plays a key role in determining which equilibrium occurs:

$$\gamma \equiv \frac{c\delta\rho - \eta\lambda(y-x)}{\delta\rho^2 + \eta\tau(\lambda-\rho)}. \tag{22}$$

In a dispersed steady state, γ is the difference between prices, $p_r - p_d$.

The key distinction in solving transitions in this model is that the differential equations are interrelated. In particular, relaxed sellers anticipate the possibility of becoming desperate, and thus their reservation price is now affected by market conditions. The resulting system of differential equations simplify to the following:

$$\dot{p}_r = (\tau+\rho)p_r - x - \tau p_d \tag{23}$$

$$\dot{p}_d = \lambda(1-\mu)(p_d - p_r) + \rho p_d - x + c \tag{24}$$

$$\dot{h}_r = \eta - \tau h_r - \lambda(1-\mu)h_r \tag{25}$$

$$\dot{h}_d = \delta + \tau h_r - \lambda h_d \tag{26}$$

These can be directly solved in relaxed or desperate phases ($\mu = 0$ or 1). For a dispersed phase, we can eliminate p_d through substitution, leaving a system of three first-order differential equations (on h_r, h_d, and p_r) which must be numerically solved.

As before, this results in a unique dispersed path,[14] with relaxed and desperate phases approaching it from above and below, respectively, as depicted in Figure 8. The dynamic features bear strong resemblance to Figure 2.B. This remains true as τ

[14]Uniqueness comes from computing the eigenvalues of the system. One is always negative while the other two are positive, indicating a saddle point. Thus, if the initial condition of this market is exactly on the saddle path (our dispersed path), it will converge to the steady state. Any other initial condition will diverge eventually, which merely indicates that one cannot sustain a dispersed path throughout the transition; rather, one must start with a relaxed or desperate path until reaching the unique dispersed path.

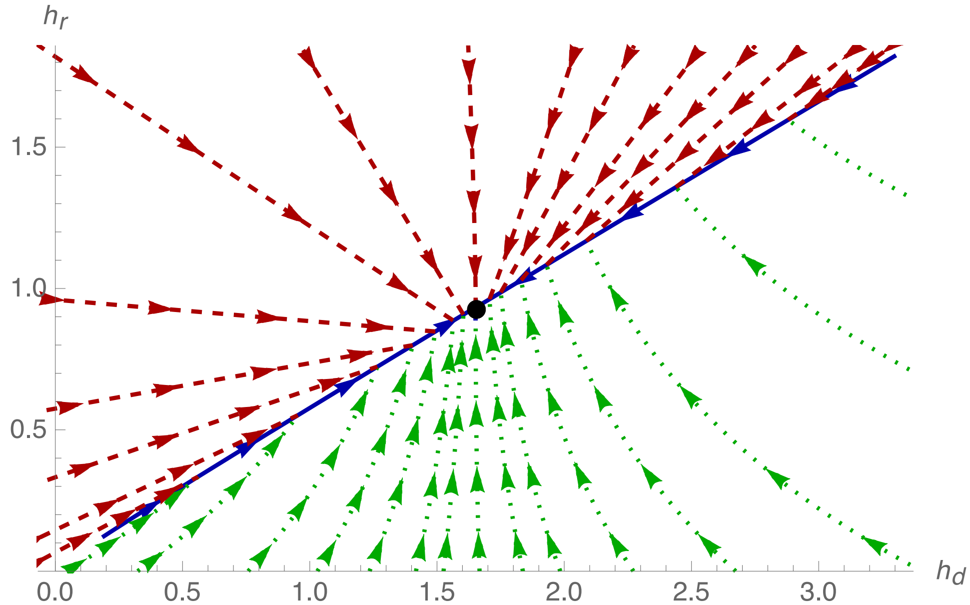

Figure 8: **Phase Portrait with Transitions to Desperate**. The solid dot indicates the dispersed steady state, while the solid line denotes the dispersed phase of the transition path. Dashed and dotted lines indicate the relaxed and desperate phases, respectively. ($y = 5.5$, $x = 5$, $c = 5$, $\eta = 1.28$, $\delta = 2.45$, $\rho = 0.1$, $\lambda = 1.5$, and $\tau = 0.03$)

becomes larger, only adding curvature to each path.

5.3 Reluctant Sellers

Within the urgency model, interesting dynamics can occur when τ is somewhat large and we let $x > y > x - c$. Literally, this means that relaxed sellers currently get a higher instantaneous flow from the asset than the buyers do. Even so, the relaxed sellers are concerned that they may soon become desperate, and are thus willing to sell the asset at a price that is below $\frac{y}{\rho}$. Thus, we refer to them as *reluctant* sellers.

In such a scenario, the dispersed path need not be unique, and within each dispersed path, the transition is no longer monotonic. Indeed, prices and populations can oscillate around their eventual steady-state values. We illustrate this in the same

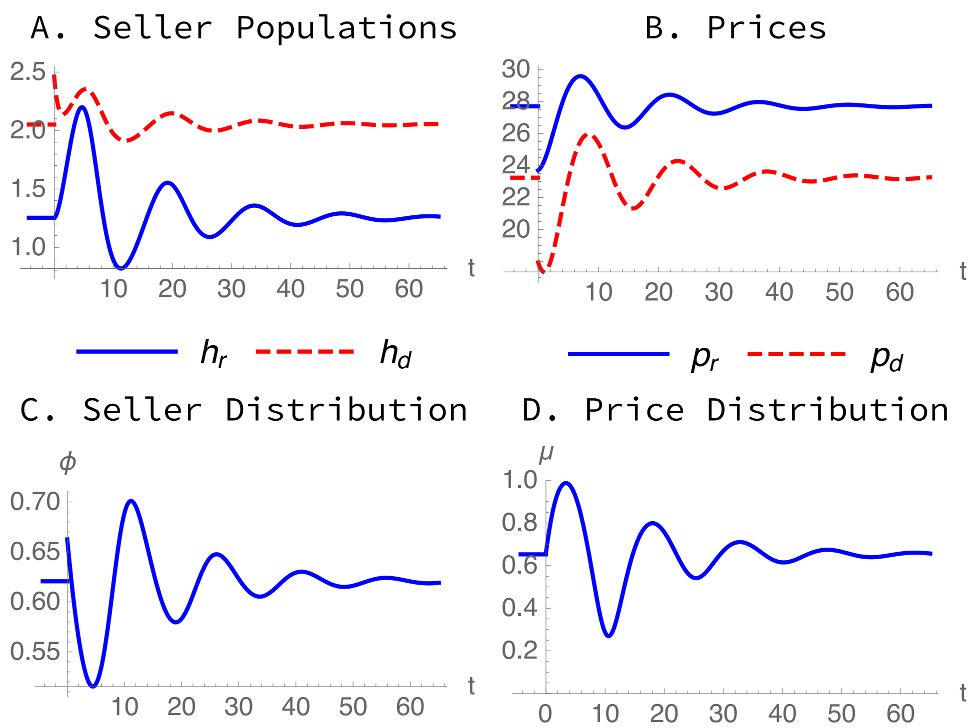

Figure 9: Dynamic response to adding 25% more desperate sellers at time $t = 0$. ($y = 3.5$, $x = 5$, $c = 5$, $\eta = 1.28$, $\delta = 2.45$, $\rho = 0.1$, $\lambda = 1.5$, and $\tau = 0.5$)

context as our first experiment. Suppose that the number of desperate sellers instantaneously increases by 20%, such as banks needing to sell foreclosed properties. Figure 9 depicts a typical dynamic response to this shock.

The remarkable feature of the equilibrium path of each variable is the gradually dampening oscillation around the steady state, rather than a monotonic adjustment. Even so, the cause of oscillation is fundamentally the same as those driving the non-monotonic transitions seen in our baseline model. The buyers respond to shocks by changing who they target; but by pursuing one type of seller more heavily, buyers unintentionally cause the other type to accumulate.

Before elaborating on this mechanism, it is helpful to note that the oscillations are nearly synchronized across certain variables. The cycles for h_r and μ are coincident, while h_d follows with a slight lag (after its first trough). The cycle for ϕ is coincident but negatively correlated with h_r.

These cycles are largely a matter of who buyers target and its effect on population dynamics. Following the shock, it is not surprising that buyers increasingly target desperate sellers. However, this causes relaxed sellers to reject more offers, raising their population by 80% within four periods. Buyers are thus attracted back to targeting relaxed sellers; yet as they do, the pool of relaxed sellers shrinks faster than that of desperate sellers, eventually making the latter more plentiful. This sets up the next cycle of oscillations, as the buyers return their focus to the desperate sellers.[15]

These population and targeting dynamics explain why prices repeatedly overshoot their steady-state levels. Prices reflect the utility of buyers anticipating future market conditions. Thus, p_r remains below steady state whenever μ is rising — fewer full price offers imply a higher chance of becoming desperate before selling the asset. As μ falls, however, relaxed sellers are more likely to obtain a full price offer, giving them a reservation price above steady state. This creates a mismatch between p_r and μ which perpetuates the cycles.

Figure 9 represents one of a continuum of equilibria, each corresponding to an initial value of $p_r(0)$. Multiple equilibria occurs here because of the interdependence of relaxed and desperate utility.[16] The reluctant sellers worry about the desperate price because they may become desperate. The desperate care about the relaxed price because they might be lucky enough to get that offer. Both types must anticipate the path of prices, so each $p_r(0)$ indicates a different anticipated path that will be

[15] The mathematical difference between this and the prior extension is that all three eigenvalues are negative around the steady state, leading to a stable spiral toward steady state. We can also combine reluctant sellers with buyer entry from our first extension with no impact on behavior. The zero profit condition again results in a constant λ throughout the dispersed transition.

[16] In the baseline model and prior extensions, the differential equations produce a saddle point at steady state. As a consequence, only one path leads directly to the steady state, while any other diverges. With reluctant sellers, however, the spiral point at steady state can be reached from an open interval of initial conditions $p_r(0)$, each creating a distinct equilibrium.

followed. Even across the multiple equilibria, the qualitative behavior is quite similar after the first cycle.

6 Conclusion

Our model depicts dynamic price formation in an economy where some sellers are more impatient than others to sell their assets. Buyers strategically choose which price they will offer, anticipating that a low price offer will be turned down by a relaxed seller who feels less pressure to liquidate his asset. This model enables us to consider the dynamic transition of seller populations, buyer strategies, and offered prices following a shock to the market.

We have three main conclusions. First, buyers almost always begin the transition by exclusively targeting one type of seller, even when the eventual steady state involves price dispersion. This seems descriptive of a fire sale, when all buyers insist on bargain prices, while any sellers capable of doing so wait for a market recovery. Second, the dynamic path of prices and seller populations often overshoot their steady state. Exclusively targeting one seller type unintentionally builds up the population of the other type; the resulting imbalance can only be relieved after both types are again targeted. Finally, in the reluctant sellers extension, prices can perpetually oscillate around steady state, slowly dampening over time.

Search frictions provide a natural reason why markets cannot immediately absorb shocks, yet they also impose discipline on the dynamic transition. Transactions are the market's only mechanism to adjust the levels or proportions of sellers, tweaking the rate at which each type exits the market. In our model, buyers effectively control this mechanism in deciding who to target, and they often find more profit from a circuitous route rather than the most direct path back to steady state.

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A Proofs

Throughout the proofs, let $F(\cdot) \equiv \int_0^1 \frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1-s+\frac{\delta}{\lambda h_d}s\right)^{\frac{1}{2}}} ds$, so that $\Delta(\hat{h}_d) = \beta \hat{h}_d + \frac{\beta\delta - \eta}{2\lambda} F(\cdot)$. We note that $F(\cdot) > 0$ because the integrand is always positive.

Proof of Lemma 1. First, consider p_r. If we substitute Eq. 6 into Eq. 5, we find that at any time, the following must hold:

$$\rho p_r = x + \dot{p}_r. \tag{27}$$

This differential equation only has one solution that converges: $p_r = \frac{x}{\rho}$ for all time.

Next, consider the h_d. The differential Eq. 7 with initial condition $h_d(0) = \frac{\delta}{\lambda} + q_d$ has $h_d(t) = \frac{\delta}{\lambda} + q_d e^{-\lambda t}$ as its unique solution.

For the h_r in either degenerate phase, the differential Eq. 8 with initial condition $h_r(0) = \frac{\eta}{\lambda} + q_r$ results in a unique solution $h_r(t) = \frac{\eta}{\lambda} + q_r + \eta t$ when $\mu = 1$ is inserted or a unique solution $h_r(t) = \frac{\eta}{\lambda} + q_r e^{-\lambda t}$ when $\mu = 0$ is inserted.

To obtain the p_d in a desperate phase, substitute $\mu = 1$ and Eq. 4 into Eq. 3 to obtain $\rho p_d = x - c + \dot{p}_d$. This differential equation has $\frac{x-c}{\rho} + a_d e^{\rho t}$ as its unique solution (up to the constant a_d). The same process in a relaxed phase ($\mu = 0$) obtains $\rho p_d = x - c + \dot{p}_d + \lambda\left(\frac{x}{\rho} - p_d\right)$, which has $\frac{x-\beta(y-x)}{\rho} + a_r e^{t(\rho+\lambda)}$ as its unique solution (up to the constant a_r).

For the dispersed phase, we first obtain $p_d(t) = \frac{x}{\rho} - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$ from rearrangement of the equal profit condition, ensuring that $\Pi(t) = 0$ throughout this phase. We then substitute this and its derivative into the pricing differential equation, $\rho p_d = x - c + \dot{p}_d + \lambda(1-\mu)\left(\frac{x}{\rho} - p_d\right)$, and solve for μ. This allows us to eliminate μ in Eq. 8, the law of motion h_r, which has the stated unique solution up to a constant. However, the solution will go to $\pm\infty$ unless the constant is set to 0, yielding the solution $h_r(t) = \Delta(h_d(t))$. □

Proof of Proposition 1. We begin by establishing that the proposed path is an equilibrium. For Case 1, suppose $h_r(0) \geq \beta h_d(0)$. From Lemma 1, we need only to establish that $\Pi(t) \leq 0$.

First, we verify that $h_r(t) \geq \beta h_d(t)$ for all t. After substituting for $h_r(t)$ and $h_d(t)$ with the proposed solutions and rearranging, this is equivalent to

$$\eta - \beta\delta \geq \lambda\left(\beta q_d - q_r\right) e^{-\lambda t}.$$

If $\beta q_d \le q_r$, then $\lambda\left(\beta q_d - q_r\right)e^{-\lambda t} \le 0$ for all t while $\eta - \beta\delta \ge 0$ by assumption, so the inequality holds. If $\beta q_d > q_r$, recall that the inequality must hold at $t = 0$ (because it is equivalent to $h_r(0) \ge \beta h_d(0)$), and $\lambda\left(\beta q_d - q_r\right)e^{-\lambda t}$ is decreasing in t. Therefore the inequality holds for all t. Thus, $h_r(t) \ge \beta h_d(t)$ for all t.

Thus, profits from offering the desperate price are always negative:

$$\Pi(t) = \frac{y-x}{\rho}\left((1+\beta)\frac{h_d(t)}{h_r(t)+h_d(t)} - 1\right) = \frac{y-x}{\rho(h_r(t)+h_d(t))}(\beta h_d(t) - h_r(t)) \le 0.$$

Moving to Case 2, suppose that $h_r(0) < \beta h_d(0)$. We first establish that there is a unique time T on the proposed path for h_d and h_r such that $h_r(T) = \beta h_d(T)$, or by substitution:

$$\frac{\eta}{\lambda} + q_r + \eta T = \beta\left(\frac{\delta}{\lambda} + q_d e^{-\lambda T}\right).$$

The l.h.s. has slope η w.r.t. T, while the r.h.s. has slope $-\beta\lambda q_d e^{-\lambda T}$. If $q_d \ge 0$, then the r.h.s. is always decreasing, creating a unique T where the two sides equate. If $q_d < 0$, then the r.h.s. is increasing and at an increasing rate $(\beta\lambda^2 q_d e^{-\lambda T})$; thus, it will eventually equate with the l.h.s. but cannot intersect again thereafter. Thus, a unique T exists such that $h_r(T) = \beta h_d(T)$.

For $t \ge T$, the Case 1 analysis applies because $h_r(t) \ge \beta h_d(t)$. We need to establish that $\Pi(t) \ge 0$ for $t < T$ in order to apply Lemma 1.

First, suppose that $q_d \ge 0$. By Lemma 1 it follows that $h_d(t) \ge \delta/\lambda$ and $h_d'(t) \le 0$ for all $t < T$. Note that the following has the same zeros and sign as the profit function:

$$\tilde{\Pi}(t) \equiv \Pi(t)\cdot\left(1 + \frac{h_r(t)}{h_d(t)}\right) = \frac{x}{\rho} - p_d(t) - \frac{y-x}{\rho}\cdot\frac{h_r(t)}{h_d(t)}.$$

The derivative of this profit function is:

$$\tilde{\Pi}'(t) = -p_d'(t) - \frac{y-x}{\rho}\cdot\frac{h_d(t)h_r'(t) - h_r(t)h_d'(t)}{h_d(t)^2} < 0.$$

The inequality holds because $p_d'(t) = \frac{\lambda c}{\rho(\rho+\lambda)}e^{\rho(t-T)} > 0$ and $h_r'(t) = \eta > 0$, while $h_d'(t) \le 0$. Since $\tilde{\Pi}(T) = 0$, this ensures that $\tilde{\Pi}(t) < 0$ for all $t < T$.

Next, suppose that $q_d < 0$, implying that $h_d(t) < \delta/\lambda$. Here, we use the following equation which has the same zeros and sign as the profit function:

$$\hat{\Pi}(t) \equiv \Pi(t) \cdot (h_d(t) + h_r(t)) = \left(\frac{y}{\rho} - p_d(t)\right) h_d(t) - \frac{y-x}{\rho}\left(h_d(t) + h_r(t)\right).$$

Using $\beta h_d(T) = h_r(T) = \frac{\eta}{\lambda} + q_r + \eta T$, we substitute for q_r to obtain $h_r(t) = \beta h_d(T) + \eta(t-T)$. Similarly, we use $h_d(T) = \frac{\delta}{\lambda} + q_d e^{-\lambda T}$ to substitute for q_d to get $h_d(t) = \frac{\delta}{\lambda} + \left(h_d(T) - \frac{\delta}{\lambda}\right) e^{\lambda(T-t)}$. We then take the inverse of this latter function, solving for t as a function of h_d and substituting for t in $\hat{\Pi}$, thereby expressing profit entirely in terms of h_d:

$$\hat{\Pi}(h_d) = \frac{c h_d \lambda^2 \left(1 - \left(\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d}\right)^{\rho/\lambda}\right) + c\lambda\rho(h_d - h_d(T)) - \eta(\lambda+\rho)(y-x)\ln\left(\frac{\delta-\lambda h_d(T)}{\delta - \lambda h_d}\right)}{\lambda\rho(\lambda+\rho)}.$$

Note that $\hat{\Pi}(h_d(T)) = 0$. Moreover, if the relaxed population is at its lowest possible level of $h_r = 0$, the corresponding desperate population on this path would be $\bar{h}_d = \frac{\delta}{\lambda} + \left(h_d(T) - \frac{\delta}{\lambda}\right) e^{\frac{\beta\lambda h_d(T)}{\eta}}$. Evaluated at that point, profit would be:

$$\hat{\Pi}(\bar{h}_d) = \frac{c}{\rho(\lambda+\rho)}\left(\rho + \lambda\left(1 - e^{-\frac{c\rho^2 h_d(T)}{\eta(\lambda+\rho)(y-x)}}\right)\right) > 0.$$

Finally, the second derivative of profit for any $h_d \in [\bar{h}_d, h_d(T)]$ is:

$$\hat{\Pi}''(h_d) = -\lambda\rho(\lambda+\rho)(\delta-\lambda h_d)^2 \left(c\rho(2\delta + (\rho-\lambda)h_d)\left(\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d}\right)^{\frac{\rho}{\lambda}} + \eta(\lambda+\rho)(y-x)\right) < 0.$$

Under the supposition that $q_d < 0$, we know that $h_d < \delta/\lambda$, so the last parenthetical term is always positive. Thus, if $\hat{\Pi}(h_d) = 0$ for some $h_d < h_d(T)$, it would have $\hat{\Pi}'(h_d) < 0$ at that h_d, and it could not later increase to reach $\hat{\Pi}(h_d(T)) = 0$. Thus $\hat{\Pi}(h_d) > 0$ for all $h_d < h_d(T)$.

We now demonstrate that the equilibrium path is unique. Lemma 1 establishes possible paths, so within that set, we verify three facts: a_r and a_d are unique, the transition must conclude in a relaxed phase, and a dispersed phase cannot occur.

First, note that if $a_r \neq 0$ then $p_d(t) \to \pm\infty$ as $t \to \infty$, which eventually violates $\Pi(t) < 0$. Similarly, in the second case, $a_d \neq \frac{\lambda c}{\rho(\rho+\lambda)} e^{-\rho T}$ is necessary to obtain $h_r(T) = \beta h_d(T)$. With a smaller a_d, buyers prefer to offer desperate prices beyond time T; with a larger a_d, buyers prefer relaxed prices before time T. Either way, buyers are making suboptimal offers and/or sellers are incorrectly anticipating the evolution of prices in the market.

Second, the equilibrium path cannot approach steady state concluding in a desperate phase. To do so, the same constant a_d is required for the desperate price to reach its steady state level (with perhaps a different T). However, any other T would violate the buyer's optimization again, as in the previous paragraph.

Finally, a dispersed phase cannot occur. When $h_r(t) < \beta h_d(t)$ or $h_r(t) > \beta h_d(t)$, buyers strictly prefer offering one price and thus cannot mix. When $h_r(t) = \beta h_d(t)$, offering either price is equally profitable, but only for an instant because h_r would grow faster than h_d. This necessarily pushes $h_r(t+\epsilon) > \beta h_d(t+\epsilon)$ for any $\epsilon > 0$, making p_d offers unprofitable the next instant. $\square$

Lemma 2. *There is a unique solution $H > 0$ to Equation 14. It always satisfies $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$. Moreover, in a dispersed phase where $h_r = \Delta(h_d)$, then $\mu(h_d) \in [0, 1)$ if $h_d > H$, $\mu(h_d) = 1$ if $h_d = H$, and $\mu(h_d) > 1$ if $h_d < H$.*

Proof of Lemma 2. The derivative of the dispersed path $h_r = \Delta(h_d)$ can be expressed as:

$$\Delta'(h_d) = \frac{\beta(\rho+\lambda)h_d^2 - \eta h_d - (\delta + (\rho-\lambda)h_d)\Delta(h_d)}{2(\lambda h_d - \delta)h_d}. \tag{28}$$

If we substitute for t as Proposition 1, the desperate path can be expressed as $h_r = \beta h_d(T) + \frac{\eta}{\lambda}\ln\frac{\delta - \lambda h_d(T)}{\delta - \lambda h_d}$. If we take its derivative w.r.t. h_d evaluated as $h_d \to h_d(T)$ (approaching the dispersed path), we obtain $\frac{\eta}{\delta - \lambda h_d}$.

The dispersed and desperate paths have the same slope when $h_r = Q(h_d) \equiv \frac{\eta(y-x) + \rho c h_d}{(y-x)(\delta + h_d(\rho-\lambda))}h_d$. Note that Eq. 14 can be written $\Delta(H) = Q(H)$.

The dispersed path has $\mu(t) = 1 - \frac{\eta - h_r'(t)}{\lambda h_r(t)}$. We note that $h_r'(t) = \Delta'(h_d(t))h_d'(t)$, and $h_d'(t) = \delta - \lambda h_d(t)$ from Eq. 7. Thus, $\mu(h_d) = 1 - \frac{\eta - \Delta'(h_d)(\delta - \lambda h_d)}{\lambda\Delta(h_d)}$. Thus, $\mu(h_d) = 1$ iff $\Delta'(h_d) = \frac{\eta}{\delta - \lambda h_d}$, or equivalently, $\Delta(h_d) = Q(h_d)$.

To show that the solution for $H > 0$ is unique, note that $\Delta(0) = 0$ and $Q(0) = 0$. However, $\Delta'(0) = \infty$ while $Q'(0) = \frac{\eta}{\delta}$, so $\Delta(\epsilon) > Q(\epsilon)$ for $\epsilon > 0$ sufficiently small. Moreover, $Q(h_d) \to +\infty$ as $h_d \to_+ \frac{\delta}{\lambda - \rho}$, but $Q(h_d) < 0$ for all $h_d > \frac{\delta}{\lambda - \rho}$. On the other hand, $\Delta(h_d) > 0$ for all $h_d > 0$. Thus, any intersection H must occur at $H < \frac{\delta}{\lambda - \rho}$.

To show that $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$, we show that $Q\left(\frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}\right) > \Delta\left(\frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}\right)$.

After substitution of $\Delta(\hat{h}_d) = \beta \hat{h}_d + \frac{\beta\delta - \eta}{2\lambda} F(\cdot)$ and rearrangement, this equivalent to:

$$(2\lambda - \rho F(\cdot))\,(\delta c - (y-x)\eta) + \eta\lambda(y-x)F(\cdot) > 0.$$

Recall that $F(\cdot) > 0$. Moreover, $\frac{(1-s)^{\frac{\rho-2\lambda}{2\lambda}}}{\left(1-s+\frac{\delta}{\lambda h_d}s\right)^{\frac{1}{2}}} \leq (1-s)^{\frac{\rho-2\lambda}{2\lambda}}$ for all $h_d \leq \frac{\delta}{\lambda}$ and all $s \in [0,1]$, so:

$$F(\cdot) \leq \int_0^1 (1-s)^{\frac{\rho-2\lambda}{2\lambda}} ds = \frac{2\lambda}{\rho}.$$

Therefore, the first parenthesis is positive and the inequality holds. Thus, the intersection of Q and Δ must occur at $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$.

Meanwhile, $Q''(h_d) = \frac{2\delta(c\delta\rho + \eta(\lambda-\rho)(y-x))}{(y-x)(\delta+(\rho-\lambda)h_d)^3} > 0$ for all $h_d < \frac{\delta}{\lambda-\rho}$. However,

$$\Delta''(h_d) = -\frac{\beta\delta - \eta}{2\lambda} \int_0^1 \frac{\delta t(1-s)^{\frac{\rho-2\lambda}{2\lambda}}(\delta s + 4h_d\lambda(1-s))}{4h_d^4\lambda^2\left(\frac{\delta s}{h_d\lambda} - s + 1\right)^{5/2}} ds < 0,$$

where the inequality holds because $s < 1$ and $\beta\delta > \eta$. Thus, $Q(H) = \Delta(H)$ exactly once for some $H > 0$.

This also means that $Q(h_d) < \Delta(h_d)$ iff $h_d < H$. Moreover, if $Q(h_d) < \Delta(h_d)$ then $\Delta'(h_d) > \frac{\eta}{\delta - \lambda h_d}$ and $\mu(h_d) > 1$. Conversely, $h_d > H$ implies $\mu(h_d) < 1$.

Finally, note that by substituting for $\Delta'(h_d)$ using Eq. 28, we find that $\mu = 1 - \frac{\eta - \Delta'(h_d)(\delta - \lambda h_d)}{\lambda\Delta(h_d)} > 0$ is equivalent to:

$$\lambda h_d \Delta(h_d)\,(\eta h_d - \delta\Delta(h_d) + h_d(\lambda+\rho)(\beta h_d - \Delta(h_d))) < 0.$$

This holds for all $h_d > 0$ since $\Delta(h_d) > \beta h_d$ and $\eta < \beta\delta$. Thus $\mu(h_d) > 0$ for all h_d.

□

Lemma 3. *If $\beta\delta > \eta$ and $h_d > H$, then $ch_d > (y-x)\Delta(h_d)$.*

Proof of Lemma 3. First, we show that $cH - (y-x)\Delta(H) > 0$. Substituting $\Delta(H) = Q(H)$, this is equivalent to:

$$\frac{c(\delta - \lambda H) - \eta(y-x)}{\delta - (\lambda - \rho)H} H > 0$$

The denominator is positive, since $H < \frac{\delta}{\lambda}$, while the numerator is positive because $H < \frac{\delta}{\lambda} - \frac{(y-x)\eta}{\lambda c}$. In addition, since $\Delta'(H) = \frac{\eta}{\delta - \lambda H}$, the derivative of $cH - (y-x)\Delta(H)$

w.r.t. H is:

$$c-(y-x)\Delta'(H)=\frac{c(\delta-\lambda H)-(y-x)\eta}{\delta-\lambda H}>0.$$

We also note that at the steady state $h_d=\frac{\delta}{\lambda}$, $ch_d-(y-x)\Delta(h_d)=\frac{(y-x)\eta}{\rho}>0$ and $c-(y-x)\Delta'(h_d)=\frac{\lambda(\rho\delta c+\eta\lambda(y-x))}{\delta\rho(2\lambda+\rho)}>0$.

Next, we demonstrate that $c>(y-x)\Delta'(h_d)$ for all $h_d\in\left(H,\frac{\delta}{\lambda}\right)$. Substituting for β and for $\Delta'(h_d)$ using Eq. 28 and rearranging:

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d>\Delta(h_d). \tag{29}$$

Since $\Delta'(h_d)>0$, $\Delta(h_d)<\Delta\left(\frac{\delta}{\lambda}\right)$ for all $h_d<\frac{\delta}{\lambda}$. Thus, Eq. 29 holds if

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d>\Delta\left(\frac{\delta}{\lambda}\right).$$

Since $\Delta\left(\frac{\delta}{\lambda}\right)=\frac{\delta c}{(y-x)\lambda}-\frac{\eta}{\rho}$, this rearranges to $h_d>\frac{\rho\delta c-\eta\lambda(y-x)}{c\rho(2\lambda-\rho)}$. Thus, $c>(y-x)\Delta'(h_d)$ for all $h_d\in\left(\frac{\rho\delta c-\eta\lambda(y-x)}{c\rho(2\lambda-\rho)},\frac{\delta}{\lambda}\right)$.

Moreover, since $\Delta(h_d)<Q(h_d)$ if $h_d\in\left(H,\frac{\delta}{\lambda}\right)$, the inequality also holds if

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d>Q(h_d)\quad\iff\quad(\delta-(\lambda-\rho)h_d)(c(\delta-\lambda h_d)-\eta(y-x))>0.$$

Thus, $c>(y-x)\Delta'(h_d)$ for all $h_d\in\left(H,\frac{\delta c-\eta\rho(y-x)}{\lambda c}\right)$.

These intervals overlap, since:

$$\frac{\rho\delta c-\eta\lambda(y-x)}{c\rho\,(2\lambda-\rho)}<\frac{\delta c-\eta\rho(y-x)}{\lambda c}\quad\iff$$

$$c\left(c\delta\rho(\lambda-\rho)+\eta\left(\lambda(\lambda-2\rho^2)+\rho^3\right)(y-x)\right)>0.$$

The inequality holds because $\lambda>2\rho$ and $\rho<1$ by assumption.

Next, consider $h_d\in\left(\frac{\delta}{\lambda},\frac{\delta}{\lambda-\rho}\right)$. In this case, the rearrangement of the derivative $c-(y-x)\Delta'(h_d)$ reverses the sign of Eq. 29:

$$\frac{c(2\delta-(2\lambda-\rho)h_d)-\eta(y-x)}{(y-x)(\delta-(\lambda-\rho)h_d)}h_d<\Delta(h_d). \tag{30}$$

The derivative of the l.h.s. w.r.t. h_d is:

$$\frac{c\left(h_d^2\rho^2 + h_d\rho(2\delta - 3h_d\lambda) + 2(\delta - h_d\lambda)^2\right) - \delta\eta(y-x)}{(y-x)(\delta + h_d(\rho-\lambda))^2}. \tag{31}$$

The derivative of this, in turn, is:

$$-\frac{2\delta(c\delta\rho + \eta(y-x)(\lambda-\rho))}{(y-x)(\delta + h_d(\rho-\lambda))^3} < 0.$$

Its denominator is positive since $h_d < \frac{\delta}{\lambda-\rho}$. The numerator is positive because $\lambda > \rho$. Thus, the largest value of Eq. 31 occurs at $h_d = \frac{\delta}{\lambda}$, which yields:

$$-\frac{c\delta\rho(\lambda-\rho) + \eta\lambda^2(y-x)}{\delta\rho^2(y-x)} < 0.$$

Thus, the l.h.s. of Eq. 30 is decreasing for all $h_d \in \left(\frac{\delta}{\lambda}, \frac{\delta}{\lambda-\rho}\right)$. We have previously established that $\Delta'(h_d) > 0$. Thus, the inequality which holds at $h_d = \frac{\delta}{\lambda}$ is relaxed further as h_d increases. Thus, $c - (y-x)\Delta'(h_d)$ for all $h_d \in \left(\frac{\delta}{\lambda}, \frac{\delta}{\lambda-\rho}\right)$.

Finally, when $h_d > \frac{\delta}{\lambda-\rho}$, the rearrangement of $c-(y-x)\Delta'(h_d)$ recovers Eq. 29. In this interval, we proceed by finding a lower bound on the l.h.s. and an upper bound on the r.h.s., then show that these former is greater than the latter. The l.h.s. is bounded below as follows:

$$\frac{c(2\delta - (2\lambda-\rho)h_d) - \eta(y-x)}{(y-x)(\delta - (\lambda-\rho)h_d)} h_d > \frac{(c(2\lambda-\rho))}{(\lambda-\rho)(y-x)} h_d.$$

which holds because this is equivalent to:

$$(\lambda-\rho)(y-x)((\lambda-\rho)h_d - \delta)(c\delta\rho + \eta(\lambda-\rho)(y-x))h_d > 0.$$

Since $\Delta''(h_d) < 0$, the r.h.s. of Eq. 29, $\Delta(h_d)$, is bounded above:

$$\Delta(h_d) < \frac{c\delta\rho(\lambda+\rho) - \eta\lambda^2(y-x)}{\delta\rho(2\lambda+\rho)(y-x)} h_d + \frac{c\delta\rho - \eta(\lambda+\rho)(y-x)}{\rho(2\lambda+\rho)(y-x)}.$$

This inequality holds because both sides are equal and share the same slope at $h_d = \frac{\delta}{\lambda}$, but the r.h.s. has a constant slope, while the slope of l.h.s. shrinks as h_d increases.

We then compare these two bounds:

$$\frac{(c(2\lambda-\rho))}{(\lambda-\rho)(y-x)}h_d > \frac{c\delta\rho(\lambda+\rho)-\eta\lambda^2(y-x)}{\delta\rho(2\lambda+\rho)(y-x)}h_d + \frac{c\delta\rho-\eta(\lambda+\rho)(y-x)}{\rho(2\lambda+\rho)(y-x)},$$

which simplifies to

$$\frac{\delta(\lambda-\rho)(c\delta\rho-\eta(\lambda+\rho)(y-x))}{\lambda^2(3c\delta\rho+\eta(\lambda-\rho)(y-x))} < h_d.$$

At the smallest $h_d = \frac{\delta}{\lambda}$, this becomes $(2\lambda+\rho)(\eta(\lambda-\rho)(y-x)+c\delta\rho) > 0$. Thus, $c-(x-y)\Delta'(h_d) > 0$ for all $h_d > \frac{\delta}{\lambda-\rho}$.

Combined, this establishes that $c-(x-y)\Delta'(h_d) > 0$ for all $h_d \geq H$. Moreover, since $cH-(x-y)\Delta(H) > 0$, this ensures that $ch_d > (x-y)\Delta(h_d)$ for all $h_d > H$. $\square$

Proof of Proposition 2. We begin by noting that the relaxed path described in Eq. 12 is found by solving $h_d = \frac{\delta}{\lambda} + \left(\hat{h}_d - \frac{\delta}{\lambda}\right)e^{-\lambda t}$ for t and substituting it into $h_r = \frac{\eta}{\lambda} + \left(\Delta(\hat{h}_d) - \frac{\eta}{\lambda}\right)e^{-\lambda t}$. The desperate path in Eq. 13 is found by similarly substituting for t in $h_r = \hat{h}_r + \eta t$.

If $h_r(0) = \Delta(h_d(0))$ and $h_d(0) \geq H$, then Lemmas 1 and 2 apply. Both prices are equally profitable throughout the dynamic transition, so the dispersed equilibrium can be maintained.

This likewise applies in the other three cases after the dispersed phase is reached. For instance, by definition, $T_r \equiv \frac{1}{\lambda}\ln\frac{\lambda q_d}{\lambda\hat{h}_d-\delta}$, which implies that $h_d(T_r) = \hat{h}_d$ per Table 2. Because $\hat{h}_d$ satisfies $h_r(0) = R\left(h_d(0), \hat{h}_d\right)$, this relaxed path ensures that $h_r(T) = \Delta(\hat{h}_d)$. Thus, $h_r(T_r) = \Delta(h_d(T_r))$ and $h_d(T_r) = \hat{h}_d \geq H$. The same reasoning applies to T_d and $\hat{h}_d$ in the third and fourth cases.

Next, as in Case 2, suppose $h_r(0) > \Delta(h_d(0))$ and $\hat{h}_d \geq H$. To show their exists a unique solution for $\hat{h}_d$, first note that $h_r(0) = R(h_d(0), \hat{h}_d)$ rearranges to:

$$\Psi(\hat{h}_d) \equiv \Delta(\hat{h}_d) - \frac{(\delta-\lambda\hat{h}_d)h_r(0)+\eta(\hat{h}_d-h_d(0))}{\delta-\lambda h_d(0)} = 0, \tag{32}$$

with a domain for Ψ of $\left[h_d(0), \frac{\delta}{\lambda}\right]$, since $\lim_{t\to\infty}\frac{\delta}{\lambda}+q_d e^{-\lambda t} = \frac{\delta}{\lambda}$. By assumption, $\Psi(h_d(0)) = \Delta(h_d(0)) - h_r(0) < 0$. On the other hand,

$$\Psi\left(\frac{\delta}{\lambda}\right) = \Delta\left(\frac{\delta}{\lambda}\right) - \frac{\eta}{\lambda} = \frac{\delta c}{(y-x)\lambda} - \frac{\eta(\rho+\lambda)}{\rho\lambda} = \beta\delta - \eta > 0.$$

Thus, by the continuity of Ψ, there exists an $\hat{h}_d$ such that $\Psi(\hat{h}_d) = 0$. Moreover, the second derivative $\Psi''(\hat{h}_d) = \Delta''(\hat{h}_d)$ is strictly negative, as shown in the proof of Lemma 2. Thus, there can be no more than two solutions for $\Psi(\hat{h}_d) = 0$. However, two distinct solutions would require $\Psi(h_d(0)) \geq 0$ and $\Psi\left(\frac{\delta}{\lambda}\right) > 0$, but this contradicts. Hence one unique solution exists for $\hat{h}_d$ and the corresponding T_r. This also assures us that $h_r(t) > \Delta(h_d(t))$ for all $t < T_r$, since $\Psi(h_d)$ only crosses zero once. In addition,

$$\Delta(h_d(t)) = \beta h_d(t) + \frac{\beta\delta - \eta}{2\lambda} F(\cdot) > \beta h_d(t),$$

for all t because $\delta\beta > \eta$ and $F(\cdot)$ is always positive.

We now show that profits are negative throughout the relaxed phase, using the rearranged profit function $\tilde{\Pi}(t) = \frac{x}{\rho} - p_d(t) - \frac{y-x}{\rho} \cdot \frac{h_r(t)}{h_d(t)}$ from Proposition 1, which shares the same signs and zeros as $\Pi(t)$. This has a first derivative of:

$$\begin{aligned} \tilde{\Pi}'(t) &= -(y-x) \cdot \left(\frac{h_r(t)}{h_d(t)}\right)' - \rho p_d'(t) \\ &= -(y-x) \cdot \frac{\eta h_d(t) - \delta h_r(t)}{h_d(t)^2} - \rho a_r(\rho+\lambda)e^{(\rho+\lambda)t}. \end{aligned}$$

Since $\Delta(\hat{h}_d) > \beta\hat{h}_d$, we know $a_r < 0$. Moreover, $h_r(t) > \beta h_d(t)$ and $\delta\beta > \eta$ imply $\eta h_d(t) < \delta h_r(t)$. Therefore $\tilde{\Pi}'(t) > 0$ for $t < T_r$. Since $\tilde{\Pi}(T_r) = 0$, then $\tilde{\Pi}(t) < 0$ and $\Pi(t) < 0$ for $t < T_r$.

Next, as in Case 3, suppose $h_r(0) < \Delta(h_d(0))$ and $h_r(0) < \frac{c-a_d}{y-x} h_d(0)$. First, we note that $\hat{h}_d$ is unique. To see this, consider the derivative $S_{\hat{h}_d} = \Delta'(\hat{h}_d) + \frac{\eta}{\lambda\hat{h}_d - \delta}$ and second derivative $S_{\hat{h}_d,\hat{h}_d} = \Delta''(\hat{h}_d) - \frac{\lambda\eta}{(\lambda\hat{h}_d-\delta)^2}$. The latter is always negative. The former is strictly positive if $\hat{h}_d > \frac{\delta}{\lambda}$, and strictly negative whenever $H < \hat{h}_d \leq \frac{\delta}{\lambda}$. Thus, S can only equal $h_r(0)$ once; indeed, $\hat{h}_d > \frac{\delta}{\lambda}$ iff $h_d(0) > \frac{\delta}{\lambda}$.

A desperate phase requires that profits are positive, which we now verify using the the rearranged profit function $\hat{\Pi}(t)$ from Proposition 1. Note that $\hat{\Pi}(T_d) = 0$ by construction. If we take the derivative w.r.t. t, we obtain:

$$\hat{\Pi}'(t) = \left(\frac{x}{\rho} - p_d(t)\right) h_d'(t) - p_d'(t) h_d(t) - \frac{y-x}{\rho} h_r'(t).$$

It is always the case that $\frac{x}{\rho} > p_d(t)$ and $h_d(t) \geq 0$. In a desperate phase, we also have $h_r'(t) = \eta > 0$. Since $a_d > 0$ by Lemma 3, we know $p_d'(t) = \rho a_d e^{\rho t} > 0$. If $h_d(t) \geq \frac{\delta}{\lambda}$, then $q_d \geq 0$ so $h_d'(t) = -\lambda q_d e^{-\lambda t} \leq 0$. Thus, $\hat{\Pi}'(t) < 0$. At $t = T_d$, profit is zero, so profit must be positive for $t < T_d$ when $h_d(0) \geq \frac{\delta}{\lambda}$.

When $h_d(0) < \frac{\delta}{\lambda}$, it is easier to examine profit in terms of the populations rather than time. After substitution, we obtain:

$$\hat{\Pi}(h_d) = \frac{\lambda h_d \left(c - \frac{c\hat{h}_d + (x-y)\Delta(\hat{h}_d)}{\hat{h}_d} \left(\frac{\delta - \lambda \hat{h}_d}{\delta - \lambda h_d} \right)^{\rho/\lambda} \right) + (y-x)\left(\eta \log\left(\frac{\delta - h_d\lambda}{\delta - \lambda \hat{h}_d} \right) - \lambda \Delta(\hat{h}_d) \right)}{\lambda \rho}.$$

Evaluated at $h_d(0)$, $\hat{\Pi}(h_d) > 0$ is equivalent to $h_r(0) < \frac{c - \rho a_d}{y-x} h_d(0)$, which holds by assumption in Case 3. We then compute the second derivative as:

$$\hat{\Pi}''(h_d) = -\frac{\lambda \eta (y-x)\hat{h}_d + \rho(2\delta + h_d(\rho - \lambda))(c\hat{h}_d - (y-x)\Delta(\hat{h}_d)) \left(\frac{\delta - \lambda \hat{h}_d}{\delta - h_d \lambda} \right)^{\rho/\lambda}}{\lambda \rho \hat{h}_d (\delta - h_d \lambda)^2} < 0.$$

The inequality holds because $h_d < \frac{\delta}{\lambda}$ and $c\hat{h}_d > (y-x)\Delta(\hat{h}_d)$ per Lemma 3. Since $\hat{\Pi}(h_d(0)) > 0$ and $\hat{\Pi}(\hat{h}_d) = 0$ while $\hat{\Pi}''(h_d) < 0$ for all $h_d \in (h_d(0), \hat{h}_d)$, profits cannot rise again after falling below zero. Therefore, $\hat{\Pi}(h_d) > 0$ for all $h_d \in (h_d(0), \hat{h}_d)$.

Moving to Case 4, the proof of Case 3 also applies to the desperate phase here, since the assumption $S(\tilde{h}_d, \hat{h}_d) = \frac{c - \rho a_d}{y-x} \tilde{h}_d$ is equivalent to $\hat{\Pi}(\tilde{h}_d) = 0$. It remains only to verify that relative profits are negative in the initial relaxed phase. This relaxed path follows a straight line segment from $(h_d(0), h_r(0))$ to $\left(\tilde{h}_d, \frac{\eta}{\lambda} + \frac{(\lambda \tilde{h}_d - \delta) q_r}{\lambda q_d} \right)$. The assumption $S(\tilde{h}_d, \hat{h}_d) = \frac{\eta}{\lambda} + \frac{(\lambda \tilde{h}_d - \delta) q_r}{\lambda q_d}$ ensures that the desperate phase continues from the latter point.

It only remains to show that profits are positive during the relaxed phase, using $\tilde{\Pi}(t)$ from before. Recall that $\tilde{\Pi}'(t) = -(y-x) \cdot \frac{\eta h_d(t) - \delta h_r(t)}{h_d(t)^2} - \rho a_r (\rho + \lambda) e^{(\rho + \lambda)t}$. Here, $\eta h_d(t) - \delta h_r(t) = (\eta h_d(0) - \delta h_r(0)) e^{\lambda t}$ by substitution. However, because $\beta \delta > \eta$, note that:

$$\begin{aligned}
0 &< \rho(\beta\delta - \eta)\left(\hat{h}_d e^{\rho T_d} + \frac{\delta F(\cdot)}{2\lambda} \right) + \beta \delta \lambda \hat{h}_d \left(e^{\rho T_d} - 1 \right) \quad \iff \\
\frac{\eta}{\delta} &< \frac{c - \frac{e^{-\rho T_d}\left(c\hat{h}_d - (y-x)\left(\beta \hat{h}_d + \frac{\beta\delta - \eta}{2\lambda} F(\cdot) \right) \right)}{\hat{h}_d}}{y-x} \quad \iff \\
\frac{\eta}{\delta} &< \frac{c - \frac{e^{-\rho T_d}(c\hat{h}_d - (y-x)\Delta(\hat{h}_d))}{\hat{h}_d}}{y-x} \quad \iff \quad \frac{\eta}{\delta} < \frac{c - a_d}{y-x}
\end{aligned}$$

Therefore, if $h_r(0) \leq \frac{\eta}{\delta} h_d(0)$, then $h_r(0) < \frac{c-a_d}{y-x} h_d(0)$, so that Case 3 would apply instead of Case 4. Thus $\delta h_r(0) > \eta h_d(0)$.

Moreover, substituting into the definition of a_r:

$$\begin{aligned} a_r &= -\frac{e^{-(\lambda+\rho)T_b}\left(\lambda c\hat{h}_d - (\lambda+\rho)\left(c\hat{h}_d - (y-x)\left(\beta\hat{h}_d + \frac{\beta\delta-\eta}{2\lambda}F(\cdot)\right)\right)e^{\rho(T_b-T_d)}\right)}{\rho(\lambda+\rho)\hat{h}_d} \\ &= -\frac{e^{-(\lambda+\rho)T_b}\left(\lambda c\hat{h}_d\left(1-e^{\rho(T_b-T_d)}\right) + \frac{(\lambda+\rho)(y-x)(\beta\delta-\eta)}{2\lambda}F(\cdot)e^{\rho(T_b-T_d)}\right)}{\rho(\lambda+\rho)\hat{h}_d} < 0. \end{aligned}$$

Hence $\tilde{\Pi}'(t) > 0$ for $t < T_b$. Since $\Pi(T_b) = 0$, therefore $\Pi(t) < 0$ for all $t < T_b$. □